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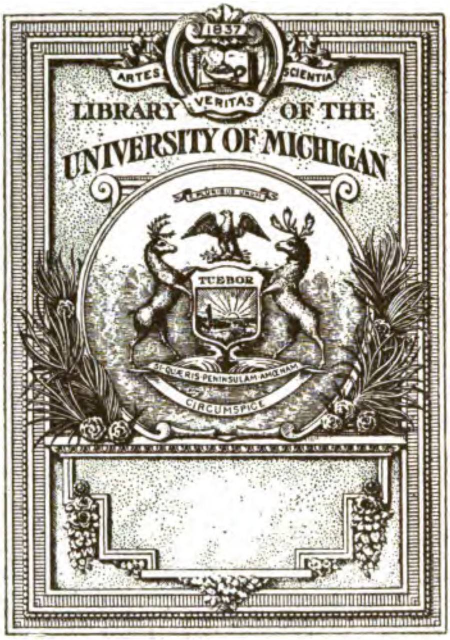
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TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY;

WITH AN

INTRODUCTION,

EXPLAINING THE

NATURE and USE of LOGARITHMS.

ADAPTED TO THE

USE OF STUDENTS IN PHILOSOPHY.

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INTRODUCTION.

ON THE NATURE AND USE OF LOGARITHMS.

Art. 1. LOGARITHMS are a ſet of artificial numbers, adapted to the common, or natural numbers, 1, 2, 3, 4, 5, &c. in order to facilitate arithmetical calculations.

2. Def. Let a be a conſtant quantity, and x be variable, and put $a^x = b$; then x is the logarithm of b.

3. Cor. Hence, if for x we write 0, 1, 2, 3, 4, 5, &c. the correſponding values of b will be a^0, a^1, a^2, a^3, a^4, a^5, &c. If therefore the natural numbers be taken in geometrical progreſſion, their reſpective logarithms will be in arithmetical progreſſion.

From this definition of logarithms, we immediately deduce thoſe properties which render them uſeful in ſhortening numerical computations.

A   4. To

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4. *To multiply numbers together, by logarithms.*

RULE. Add the logarithms of the numbers together, and the ſum is the logarithm of their product.

For let $a^x = b$, $a^y = c$, $a^z = d$; then, $a^{x+y+z} = bcd$; and (2 *) x is the logarithm of b, y the logarithm of c, z the logarithm of d, and $x+y+z$ is the logarithm of bcd.

5. *To divide one number by another, by logarithms.*

RULE. From the logarithm of the dividend, ſubtract the logarithm of the diviſor, and the remainder is the logarithm of the quotient.

For let $a^x = b$, $a^y = c$; then $\frac{b}{c} = \frac{a^x}{a^y} = a^{x-y}$; and (2) x is the logarithm of b, y the logarithm of c, and $x-y$ is the logarithm of $\frac{b}{c}$.

6. *To raiſe a number to any power, by logarithms.*

RULE. Multiply the logarithm of the root by the index of the power, and the product is the logarithm of the power.

For let $a^x = b$; then $a^{rx} = b^r$; and (2) x is the logarithm of b, and rx is the logarithm of b^r.

7. *To extract the root of any number, by logarithms.*

RULE. Divide the logarithm of the given number, by the number which expreſſes the root to be extracted, and the quotient is the logarithm of the root.

For let $a^x = b$; then $a^{\frac{x}{r}} = b^{\frac{1}{r}}$; and (2) x is the logarithm of b, and $\frac{x}{r}$ is the logarithm of $b^{\frac{1}{r}}$.

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* The figures in a parentheſis refer to the Articles.

Thus it appears, that multiplication may be performed by the addition of logarithms; diviſion, by the ſubtraction of logarithms; the raiſing of powers, by the multiplication of a logarithm; and the extracting of roots, by the diviſion of a logarithm.

8. As a may be aſſumed any number, we may have the ſame ſet of logarithms adapted to different ſets of natural numbers; and the ſame ſet of natural numbers may have different ſets of logarithms. But as $a^0 = 1$, in every ſyſtem the logarithm of 1 is 0.

9. The Tables of logarithms in common uſe, are conſtructed upon ſuppoſition that $a = 10$. Now $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, &c. hence, in this ſyſtem, the log. of 1 is 0; the log. of 10 is 1; the log. of 100 is 2; the log. of 1000 is 3; the log. of 10000 is 4; &c. and, in general, if $10^x = b$, x is the logarithm of b. Hence, the logarithms of all numbers from 1 to 10 are expreſſed by 0, with decimals; from 10 to 100, they are expreſſed by 1, with decimals; from 100 to 1000, they are expreſſed by 2, with decimals; from 1000 to 10000, they are expreſſed by 3, with decimals; &c. Theſe numbers 0, 1, 2, 3, &c. are called indices, and the index is always leſs by unity than the number of figures in the integral part of the correſponding natural number. One method of finding the logarithms of the intermediate numbers between 1 and 10, 10 and 100, 100 and 1000, &c. has been from the equation $10^x = b$, by finding x, having given b. But it is not our intention to ſhow the methods of computing logarithms.

10. The logarithm of 76348 is 4,8827972; for $10^{4,8827972} = 76348$; divide by 10, and we get $10^{3,8827972} = 7634,8$; divide again by 10, and we get

$10^{2,8827972} = 763,48$; and ſo on. Hence, if any number be multiplied or divided by 10, the index of it's logarithm will be increaſed or diminiſhed by 1, the decimal part remaining the ſame. If therefore we continue to divide by 10 till the number becomes leſs than unity, and the index be negative, the decimal part is ſtill poſitive; on which account, the negative ſign is put *over* the index, becauſe that is the only figure affected by it. When therefore the negative ſign is put *over* the index, it is to be underſtood that the index is negative, and the decimal part poſitive; but when the negative ſign is put *before* the index, it is here to be underſtood, that both index and decimal part are *negative*.

11. If the ſeries be continued downwards below 1, as 10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}, &c. the reſpective logarithms are 3, 2, 1, 0, -1, -2, -3, &c. Therefore the logarithm of any number which ſtands between any two terms of the firſt ſeries, has for it's index the index of the leaſt of thoſe two terms, together with a decimal which is always poſitive, unleſs otherwiſe ſpecified.

Thus, the logarithm of any number between

10^3 and 10^2 is 2, + dec.
10^2 and 10^1 is 1, + dec.
10^1 and 10^0 is 0, + dec.
10^0 and 10^{-1} is $\bar{1}$, + dec.
10^{-1} and 10^{-2} is $\bar{2}$, + dec.
10^{-2} and 10^{-3} is $\bar{3}$, + dec.
&c. &c.

The logarithm thus expreſſed is the *true* logarithm.

12. Sometimes, inſtead of writing the index negative, 10 is added to it, and the index becomes poſitive; in this caſe, the logarithm of any number between

10^{0} and 10^{-1} is $9, +$dec.
10^{-1} and 10^{-2} is $8, +$dec.
10^{-2} and 10^{-3} is $7, +$dec.
&c. &c.

13. But here, when we come to 10^{-10}, the index becomes negative; in this caſe therefore, to preſerve the index poſitive, it is uſual to add 100 to the index; hence, the logarithm of any number between

10^{0} and 10^{-1} is $99, +$dec.
10^{-1} and 10^{-2} is $98, +$dec.
10^{-2} and 10^{-3} is $97, +$dec.
&c. &c.

14. Hence, we have the following gradation of numbers and their logarithms, the logarithms of decimal numbers being written each way.

Number.	*True* Logarithm	Logarithm of decimal numbers, adding 10 to the index.	Logarithm of decimal numbers, adding 100 to the index.
76348	4,8827972		
7634,8	3,8827972		
763,48	2,8827972		
76,348	1,8827972		
7,6348	0,8827972		
,76348	$\bar{1}$,8827972	9,8827972	99,8827972
,076348	$\bar{2}$,8827972	8,8827972	98,8827972
,0076348	$\bar{3}$,8827972	7,8827972	97,8827972
,00076348	$\bar{4}$,8827972	6,8827972	96,8827972

The negative index denotes how many places the firſt ſignificant figure of the decimal number is below unity; and when 10, or 100 is added to the index, how much ſuch index wants of 10, or 100, denotes the ſame. The negative index, however, is that which ſtands in the regular ſcale of logarithms, and always repreſents the *true* logarithm of a decimal, and of that one number only; whereas, the logarithm of a decimal, expreſſed by adding 10, or 100, to the index, is 10, or 100 too great, and expreſſes alſo the logarithm of a number greater than unity. Thus, 4,8827972 is the logarithm of 76348; and conſidering it as the logarithm of a decimal (12), it is alſo the logarithm of 0,0000076348. By uſing the negative index, there is no danger of a miſtake, and every ſource of error ſhould be cut off; we ſhall therefore derive all our concluſions in terms of the *true* logarithm.

15. At the end of any operation where the logarithm of a decimal is taken by adding 10 to the

index, and that logarithm is *added*, the 10 added muſt be *ſubtracted*; and if there be any number (r) of ſuch logarithms *added*, then 10r muſt be *ſubtracted*, and the reſult will be the *true* logarithm. But if a logarithm of this kind is to be *ſubtracted*, then 10 muſt be *added* to the reſulting index, to give the *true* logarithm; for by ſubtracting a quantity which is 10 above it's true value, you make the reſult 10 leſs than it's true value, and therefore 10 muſt be added to give the *true* logarithm. In like manner, if you add 100 to the index, you muſt, in the former caſe, *ſubtract* 100 from the index for every ſuch logarithm *added*; and in the latter caſe, you muſt *add* 100, for the logarithm which is *ſubtracted*; and the reſult will be the *true* logarithm.

16. *To find the logarithm of a number conſiſting of* one, two, three, *or* four *figures.*

Rule. The logarithms of all numbers to 100 are put down with their proper indices; and from 100 the logarithms are put down without their indices; and when the logarithm is taken out, it is the decimal part only of the logarithm, and the index is ſupplied, being always leſs by unity than the number of figures in the integral part of the given number (9). From 100 to 1000, the decimal part of the logarithm ſtands in the column (Log.) directly againſt the correſponding number in the column (N); and as the two firſt figures are common to ſeveral logarithms, they are not repeated till they alter; they are therefore to be prefixed to the five others in the lines below them, till you come to the next two common figures. The decimal part of the logarithm of a number conſiſting of *four* figures is found in the ſecond column under o at the head, directly againſt the number in the firſt column. And in taking out the logarithm correſponding to any number conſiſting of four or more figures, the three firſt figures of the decimal part of the logarithm are found in the ſecond column, and theſe are common to all the figures in that and in the other nine columns, till you come to the next three figures; except when the firſt figure on the left of the other four figures, found in that or in the other nine columns, changes from 9 to 0, in which caſe, the next three figures are to be prefixed to the four figures thus beginning with 0, and to each of the four following figures in the ſame horizontal line. For example: the decimal part of the logarithm of 30837 is 4890721; for though againſt 3083 the three common figures next above are 488, yet from the third to the fourth column, the firſt of the next four figures changes from 9 to 0, therefore we muſt take 489 for the three firſt figures.

17. *To find the logarithm of a number having* five *figures.*

RULE. Find the firſt four figures of the given number in the firſt column of the Table, and even with them in the ſecond column, take out the three firſt common figures; go on in the ſame horizontal line till you come to the column which has at it's head the fifth figure of the given number, and take out the four figures there found, and annex them to the right of the three before found, and you have the decimal part of the logarithm; and the index is the number which is leſs by unity than the number of integral figures in the given number (9).

Ex. *Let the given number be* 39217.

Here, in the ſecond column againſt 3921, you find 593 the three firſt figures, and in the ſame horizontal line under 7, you have 4744; therefore the required logarithm is 4,5934744.

18. *To find the logarithm of a number having* ſix *figures.*

Rule. Find the decimal part of the logarithm for the firſt five figures (17), and take the difference (*d*) between that logarithm and the logarithm next greater; then find that difference in the laſt column but one marked *D*, and under that difference in the laſt column marked *Pts* (proportional parts) againſt the figure in the ſixth place, you have the part to be added for that figure.

Ex. *Let the given number be* 392176.

The decimal part of the logarithm for the firſt five figures is 5934744, the difference between which and the next greater logarithm is 110; and in the laſt column under 110, and againſt 6, you find 66; therefore the decimal part of the logarithm is 5934744 + 66 = 5934810; and there being ſix places of integral figures, the required logarithm is 5,5934810.

19. *To find the logarithm of a number having* ſeven *figures.*

Rule. Find the decimal part of the logarithm for the firſt ſix figures (18); then divide the number in that part of the column *Pts* having (*d*) at it's head correſponding to the ſeventh figure, by 10, and add it to the decimal part of the logarithm for the firſt ſix figures; remembering to place the firſt figure on the right of the quantity added, in the eighth place of the logarithm.

Ex. *Let the given number be* 3921764.

The decimal part of the logarithm for the firſt ſix figures is 5934810; and in the laſt column under 110, 44 ſtands againſt 4, the tenth part of which is 4,4; therefore the decimal part of the logarithm is 59348144; and there being ſeven places of whole numbers, the required logarithm is 6,59348144.

20. *To find the logarithm of a number having* eight *figures*

Rule. Find the decimal part of the logarithm for the firſt ſeven figures (19); then divide the number in that part of the column *Pts* having (*d*) at it's head correſponding to the eighth figure, by 100, and it will give you the number to be added to the decimal part of the logarithm, for that figure; remembering that if the quotient conſiſt of an integer and decimals, the integer muſt ſtand in the ſeventh place; and if there be only decimals, the firſt decimal on the left muſt ſtand in the eighth place.

Ex. *Let the given number be* 39217648.

The decimal part of the logarithm for the firſt ſeven figures is 59348144; and in the laſt column under 110, 88 ſtands againſt 8, which divided by 100, gives 0,88; hence, the decimal part of the logarithm is 593481528; and there being eight places of whole numbers, the required logarithm is 7,593481528.

21. If the given number be not all integral, the decimal part of the logarithm is found in the ſame manner, and the index is leſs by unity than the number of figures in the integral part (9).

Ex. *Let the given number be* 392,1764.

Here, the decimal part of the logarithm is 5934814; hence, the logarithm required is 2,5934814.

22. *To find the logarithm of a decimal.*

Rule. Omitting the decimal cyphers on the left, if there be any, find the logarithm correſponding to the other part, as for a whole number, and (14) annex a negative index, which expreſſes how far the firſt ſignificant figure of the decimal, is from the place of units. Or, if you write the logarithm with a poſitive index, the index will be as much below 9, or 99, as there are decimal cyphers on the left.

Ex. *To find the logarithm of* 0,0004346.

Look into the Table, and againſt 4346, you will find the logarithm 6380897; and there being three decimal cyphers on the left of the ſignificant figures, the required logarithm becomes $\bar{4},6380897$, if expreſſed by a negative index; or 6,6380897, if expreſſed by a poſitive index.

23. *To find the number to* five *figures anſwering to any given logarithm.*

Rule. Neglect the index, and ſeek for the decimal part of the logarithm, the three firſt figures in the ſecond column, and the other four in any of the columns, and even with theſe four figures in the firſt column *N*, you have the firſt four figures of the required number; and the fifth figure is that which ſtands at the head of the column where the four figures were found; and the number of integral figures is greater by unity than the index (9).

Ex. *Let the given logarithm be* 2,4967775.

Having found the three firſt decimal figures 496 in the ſecond column, againſt 7775 you find 3138 in the firſt column, and at the head of 7775 you find 9, which annexed to 3138 make 31389; and the index being 2, the required number is 313,89.

If the logarithm be found exactly, and the index be greater than 4, cyphers muſt be added to the right of the five figures, till the whole number of figures be greater by unity than the index; for by increaſing the index by 1, you make the correſponding number ten times greater (10).

24. *To find the number to* ſix *or* ſeven *figures, anſwering to any given logarithm.*

Rule. The given logarithm (A) not being found exactly in the Table, take the next leſs (B), and in the column *Pts*, having at it's head the difference between the logarithm B and the next greater logarithm, look for $A-B$, and againſt it you have the *ſixth* figure; but if $A-B$ be not found exactly, take the figure in the column *Pts* correſponding to the next leſs number than $A-B$; for the *ſixth* figure required; then take the difference between $A-B$ and the next leſs number in the column *Pts*, and prefix a cypher to the right, and againſt that number, or the neareſt to it, you have the *ſeventh* figure required; and the number of integral figures is greater by unity than the index.

Ex. *What is the number correſponding to the logarithm* 6,4970385?

The difference between the logarithms next greater and leſs than the given logarithm, is 139; this number therefore ſtands at the head of the column *Pts* to be entered.

Given log. - - 6,4970385 = *A*
Next leſs log. - - 6,4970264 = *B*

121 = *A* − *B*
8 is the *ſixth* figure - - 111 next leſs.

7 is the *ſeventh* figure - - 100

Now the firſt five figures of the number anſwering to *B*, are 31407; therefore the required number is 3140787.

25. Or the additional figures, as many as may be required, may be found thus. Add as many cyphers to the right of *A* − *B* as you want additional figures, and divide that quantity by the difference between the logarithm *B* and the next greater logarithm, and the quotient gives the figures required.

In the laſt example, if we add two cyphers to *A* − *B*, it becomes 12100, which divided by 139 (the difference of the logarithm *B* and the next greater logarithm,) the quotient is 87 the two next figures.

26. If the given logarithm be the logarithm of a decimal with a poſitive index (12), find the number correſponding to the decimal part, and prefix as many decimal cyphers to the left of it, as the index wants of 9 (14).

Ex. *Let the logarithm be* 6,6380897, *and let it repreſent the logarithm of a decimal* with a poſitive index.*

Look into the Table, and againſt the log. 6380897 you find the number 4346; and the index 6 being 3 leſs than 9, there muſt be three cyphers added to the left; hence, the required decimal number is 0,0004346.

27. If the logarithm be written with a negative index, as $\overline{4},6380897$, find the ſignificant figures of the correſponding number as before, and the index ſhows how many places the firſt ſignificant figure is below unity.

28. *To find the logarithm of a proper fraction.*

Rule. Add 10 to the index of the logarithm of the numerator, and ſubtract the logarithm of the denominator from it, and you get the logarithm of the fraction, expreſſed by a poſitive index, having 10 added to the index, as in decimals. Or, ſubtract the logarithm of the denominator from that of the numerator, and you get the logarithm of the fraction, expreſſed by a negative index.

Ex. *To find the logarithm of* $\frac{3}{5}$.

Here, by the Rule, the logarithm is 10, + log. 3 − log. 5 = 10,4771213 − 0,6989700 = 9,7781513. Or, the required logarithm is log. 3 − log. 5 = 0,4771213 − 0,6989700 = $\overline{1},7781513$.

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* It may alſo repreſent the logarithm of a whole number having ſeven figures.

29. When it is required to incorporate ſeveral logarithms by addition and ſubtraction, it will be more convenient to convert the ſubtraction into an addition, by writing down, inſtead of the logarithm to be ſubtracted, what it wants of 10,0000000, which you may very readily do, by writing down what the firſt figure on the right wants of 10, and what every other figure wants of 9; this is called the *Arithmetic Complement*. For inſtance, if the logarithm be 7,4693875, it's arithmetic complement is 2,5306125. If one or more figures to the right be cyphers, write cyphers in their place, and take the firſt ſignificant figure on the right, from 10, and the other figures from 9. Thus, if the logarithm be 5,3864000, it's arithmetic complement is 4,6136000. In any operation therefore, having taken the arithmetic complements of all the quantities to be ſubtracted, the number of which we will ſuppoſe to be r, and added the whole together, you muſt ſubtract $10r$ from the index. The reaſon of this is evident; for to add what a number wants of 10, muſt evidently make a quantity greater by 10, than if you had ſubtracted that number. For inſtance; $14+6$ is greater by 10 than $14-4$. If the index of the quantity whoſe arithmetic complement you want to take, be greater than 10, write down what the index wants of 20, and after the addition, ſubtract 20 from the index.

30. When you have a logarithm with a negative index, it will ſometimes be convenient to reduce the whole to a negative quantity; and this is done, by ſubtracting 1 from the index, taking the arithmetic complement of the decimal part, and prefixing the ſign − *before* the index, ſo as to affect the whole; for by this

operation, you increaſe the value of the index by unity, and diminiſh that of the decimal part by unity, and therefore the value of the logarithm is not altered. Thus, $\bar{3}$,5972684 = − 2,4027316; theſe we ſhall call *negative* logarithms. This reduction will often be found convenient when you want to multiply or divide a logarithm with a negative index, for otherwiſe, you muſt, in general, multiply or divide the index, and the decimal part, ſeparately. And here, the reſulting quantity muſt be reduced back again to it's negative index, and poſitive decimal, in order to render it conformable to the ſcale of logarithms in the Tables; and this is done, by adding 1 to the index, and taking the arithmetic complement of the decimal part. Thus, − 3,5946281 = $\bar{4}$,4053719. The above reduction to a *negative* logarithm, if you want to ſubtract ſuch a logarithm, is very convenient, for you have then only to add the logarithm after you have reduced it. Thus, if you want to ſubtract $\bar{3}$,5972648, you may add, 2,4027316, and the operation is performed; for the ſubtraction of a negative quantity, is the ſame as the addition of a poſitive quantity of the ſame magnitude.

31. When you have to ſubtract the logarithm of a decimal, and you write the logarithm by adding 10 to the index, you may take it's arithmetic complement and add it, and it gives the true reſult. For if you add the arithmetic complement, it makes the reſult greater by 10, than if you had ſubtracted the logarithm itſelf; but as the logarithm is 10 too great, by ſubtracting it, the reſult becomes 10 too little; therefore by adding the arithmetic complement, you get the true reſult.

Having explained the principles of logarithms, we proceed to exemplify their uſe in numerical calculations.

32. *To multiply numbers together.*

RULE. If the numbers be greater than unity, add their logarithms together, and the ſum is the logarithm of the product (4).

Ex. 1. *What is the product of* 2,71 × 38 × 4,9?

0,4329693	log. of - - - -	2,71
1,5797836	—— - - -	38
1,6901961	—— - - -	4,9
3,7029490	—— - -	5046,02 Product.

33. If any of the numbers be below unity, the rule is the ſame, if you write their logarithms with the negative index; for then you have the *true* logarithm.

Ex. 2. *What is the product of* 784 × 0,000079 × 0,0000036?

2,8943161	log. of - - -	784
$\bar{5}$,8976271	—— - -	0,000079
$\bar{6}$,5563025	—— - -	0,0000036
$\bar{7}$,3482457	——	0,0000002229696 Product.

Here, we carry 2 from the decimal part of the logarithm (which is poſitive) to the index, which added to 2 (the firſt index) make 4 for the poſitive part of the index, and the negative part is 11; therefore the ſum is $4 - 11 = - 7$; and (14) there are ſix cyphers before the firſt ſignificant figure.

34. If the operation be performed by adding 10 to the index of the logarithm of the decimal factors, and there be *r* ſuch factors, you muſt ſubtract 10*r* from the index after the addition (15). We will take the laſt example.

2,8943161 log. of - - - - - - 784

5,8976271 ——— - - - - 0,000079

4,5563025 ——— - - - - 0,0000036

$\overline{7}$,3482457 the ſame as before.

Here, the index of the ſum is 13, and $r = 2$, therefore $10r = 20$; hence, the index is $13 - 20 = - 7$.

35. If the numbers be ſome, or all of them, leſs than unity, the product may be found by the following

Rule. Conſider the ſignificant figures of each factor, as whole numbers; add together their logarithms; and ſubtract as many units from the index, as there are decimal figures in all the numbers, and you have the *true* logarithm of the product.

For let the given numbers be *a*, *b*, *c*, &c. and ſuppoſe them to contain *n* decimals; and let *A*, *B*, *C*, &c. be their reſpective values, conceiving the ſignificant figures to repreſent whole numbers; then, by the nature of decimals, $a \times b \times c \times$ &c. $= \frac{A \times B \times C \times \&c.}{10^n}$; hence (4, 5), the logarithm of $a \times b \times c \times$ &c. = log. A + log. B + log. C + &c. − n*.

Ex. 1. *What is the product of* 2,4 × 0,007 × 0,54?

Here, $n = 6$; hence,

1,3802112	log. of	24	
0,8450980	——	7	
1,7323938	——	54	
$\bar{3}$,9577030	——	0,009072	Product.

Here, the index of the ſum is 3, from which ſubtract 6 (n), and the remainder is − 3; therefore the firſt ſignificant figure is 3 places below unity (14).

Ex. 2. *What is the product of* 784 × 0,000079 × 0,0000036?

Here, $n = 13$; hence,

2,8943161	log. of	784	
1,8976271	——	79	
1,5563025	——	36	
$\bar{7}$,3482457	——	0,0000002229696	Product.

---

* For the log. of 10^n is n (9).

Here, the index of the ſum is 6, from which ſubtract 13 (*n*), and the remainder is − 7; therefore the firſt ſignificant figure is 7 places below unity (14).

Thus, the product of decimal factors as well as of whole numbers, may be always found by taking the logarithms of whole numbers only.

36. *To divide one number by another.*

Rule. From the logarithm of the dividend ſubtract the logarithm of the diviſor, and the remainder is the logarithm of the quotient (5).

Ex. 1. *Divide* 2004,64 *by* 34.

3,3020364	log. of - -	2004,64
1,5314789	—— - -	34
1,7705575	—— - -	58,96 Quotient.

Ex. 2. *Divide* 19 *by* 72.

1,2787536	log. of - - -	19
1,8573325	—— - - -	72
$\bar{1}$,4214211	—— - -	0,2638889 Quotient.

Here, as we carry 1 from the decimal to the index, we have to ſubtract 2 in the index from 1, the remainder is therefore − 1, which ſhows that the firſt ſignificant figure of the quotient is 1 place below unity (14).

In all caſes the operation is the ſame, if the negative index be uſed for the logarithms of all quantities below unity.

37. If the dividend be leſs, and the diviſor be greater than unity, and you write the logarithm of the former by adding 10 to the index, you muſt add 10 to the index of the logarithm of the latter, before you ſubtract. For making each index equally too great, you get the true difference.

Ex. '*Divide* 0,000084 *by* 714.

By adding 10 to the index.

5,9242793 log. of - - 0,000084
2,8536982 ——— - - - 714

$\bar{7}$,0705811 ——— 0,000000117647 Quotient.

Here, 10 added to 2 make 12, which taken from 5, the remainder is − 7.

By the negative index,

$\bar{5}$,9242793 log. of - - 0,000084
2,8536982 ——— - - - 714

$\bar{7}$,0705811 ——— 0,000000117647 Quotient.

38. If the dividend be greater, and the diviſor leſs than unity, and you write the logarithm of the latter by adding 10 to the index, you muſt add 10 to the index of the logarithm of the former, before you ſubtract. This is true, for the reaſon before given.

Ex. *Divide* 96 *by* 0,00042.

By adding 10 to the index.

1,9822712	log. of - - -	96
6,6232493	—— - - -	0,00042
5,3590219	—— - - -	228571,4 Quotient.

Here, 10 added to 1 make 11, from which ſubtract 6, and the remainder is 5.

By the negative index.

1,9822712	log. of - - -	96
$\bar{4}$,6232493	—— - -	0,00042
5,3590219	—— - -	228571,4 Quotient.

Here, – 4 ſubtracted from 1, the remainder is 5.

39. If the diviſor and dividend be both leſs than unity, and their logarithms be written by adding 10 to the index, the ſubtraction gives the true reſult. For the true index of each logarithm being equally increaſed, their difference is the true difference.

Ex. *Divide* 0,2 *by* 0,00057,

By adding 10 to the index.

9,3010300	log. of - - -	0,2
6,7558749	—— - -	0,00057
2,5451551	—— - -	350,877 Quotient.

By the negative index.

$\bar{1},3010300$ log. of - - - 0,2
$\bar{4},7558749$ —— - - 0,000057

$2,5451551$ —— - - 350,877 Quotient.

Here, carrying 1 to -4, the ſum is -3, which ſubtracted from -1, the remainder is 2.

The operation by the negative index is preferable to that by the other method, as there is no variety of caſes.

40. If the diviſor and dividend conſiſt of factors, of which ſome, or all, are decimal numbers, the quotient may be found by the following

Rule. Let the dividend be $a \times b \times c \times$ &c. containing n decimals in all the factors together; and let the diviſor be $s \times t \times u \times$ &c. to r factors, containing m decimals in all the factors; and let A, B, C, &c. and S, T, U, &c. be the reſpective values of a, b, c, &c. and s, t, u, &c. conſidering their ſignificant figures as whole numbers; then the logarithm of the quotient = log. A + log. B + log. C + &c. + ar. co. log. S + ar. co. log. T + ar. co. log. U + &c. + $m - n - 10r$.

For, by the nature of decimals, $\frac{A \times B \times C \times \&c.}{10^n} = a \times b \times c \times$ &c. and $\frac{S \times T \times U \times \&c.}{10^m} = s \times t \times u \times$ &c.

hence, $10^{m-n} \times \frac{A \times B \times C \times \&c.}{S \times T \times U \times \&c.} = \frac{a \times b \times c \times \&c.}{s \times t \times u \times \&c.}$; therefore (4, 5, 29) the log. of the required quotient $=$ log. A + log. B + log. C + &c. + ar. co. log. S + ar. co. log. T + ar. co. U + &c. $+ m - n - 10r$.

Ex. *What is the value of* $\frac{84 \times 0{,}00769 \times 0{,}683}{598 \times 0{,}0000146 \times 0{,}039}$?

Here, $n = 8$, $m = 10$, $m - n = 2$, $r = 3$, $10r = 30$, and $m - n - 10r = -28$; hence,

1,9242793	log. of - - - -	84
2,8859263	—— - - - -	769
2,8344207	—— - - - -	683
7,2232988	ar. co. log. of - -	598
7,8356471	—— - -	146
8,4089354	—— - -	39
3,1125076	- - - - - -	1295,71 Quotient.

The index of the ſum of the logarithms is 31, from which ſubtract 28, and the remainder is 3.

By this method, the operation is extremely plain and eaſy, and you have to take the logarithms of whole numbers only.

41. *To raiſe numbers to any power.*

Rule. Multiply the logarithm of the root by the index of the power, and the product is the logarithm of the power (6).

Ex. *What is the 16th power of 1,05?*

0,0211893 log. of 1,05
16

0,3390288 —— 2,182875 the power required.

42. If the number to be raiſed to a power be leſs than unity, and you expreſs it's logarithm by a negative index, the operation is performed by the following

Rule. Multiply the index of the logarithm, and the decimal part, ſeparately by the index of the power, and ſubtract the former product (it being negative) from the latter, and the remainder is the logarithm of the power.

For the index being negative, and the decimal part poſitive, the logarithm, as it ſo ſtands, cannot generally be multiplied as one quantity; the index and logarithm are therefore multiplied ſeparately, and the former product is taken from the latter.

Ex. 1. *What is the* 8,3 *power of* 0,042?

The log. of 0,042 is $\bar{2}$,6232493; hence,

,6232493
8,3

18697479
49859944

From 5,17296919
Sub. 16,6 = 8,3 × 2

Rem. $\overline{12}$,57296919 log. of 0,0000000000037408, the power required.

Ex. 2. *What is the* 0,07 *power of* 0,00563?

The log. of 0,00563 is $\bar{3}$,7505084; hence,

,7505084
0,07

From 0,052535588
Sub. 0,21 = 0,07 × 3

Rem. $\bar{1}$,842535588 log. of 0,695882 the power required.

43. If you reduce the logarithm of the root to a *negative* logarithm (30), the operation may be performed by one multiplication. Let us take the caſe of the laſt example.

The logarithm $\bar{3},7505084 = -2,2494916$; hence,

$$\begin{array}{r} -2,2494916 \\ 0,07 \\ \hline -0,157464412 \end{array} = \bar{1},842535588$$

the ſame as before.

The other method will generally be found moſt ready in practice, as you have here to reduce the given logarithm to a *negative* logarithm, and then at laſt to reduce it back again. This method, however, will be found very uſeful when you have to find the value of a decimal number having an index expreſſed by a vulgar fraction.

44. If the number to be raiſed to a power be leſs than unity, and you expreſs it's logarithm by adding 10 to the index, the operation may be performed by the following

Rule. Let n be the index of the power; multiply the logarithm of the root by n, and from the product ſubtract $10n$, and the remainder is the *true* logarithm of the power.

For the logarithm of the root being 10 greater than the true logarithm, when you multiply it by n, the logarithm muſt be $10n$ greater than the true logarithm, and therefore you muſt ſubtract $10n$ from the product, in order to get the true logarithm of the power.

Ex. 1. *What is the* 8,3 *power of* 0,042?

8,6232493 log. of 0,042
8,3

258697479
689859944

From 71,57296919
Sub. 83, $= 10 \times 8{,}3$

Rem. $\overline{12}$,57296919 log. of 0,0000000000037408

the power required.

Ex. 2. *What is the* 0,07 *power of* 0,00563?

7,7505084 log. of 0,00563
0,07

From 0,542535588
Sub. 0,7 $= 10 \times 0{,}07$

Rem. $\overline{1}$,842535588 log. of 0,695882 the power required.

45. It will ſometimes happen, that the logarithm produced will not be the exact logarithm of the number required. For inſtance; the log. of 7^5 is $5 \times$ log. $7 = 5 \times 0{,}8450980 = 4{,}2254900$. Now $7^5 = 16807$, the log. of which is 4,2254902, differing from the above log. by 2 in the laſt figure; and this ariſes from the log. of every number being only an approximation, on which account, the error, when

multiplied, becomes manifeſt. Had the firſt logarithm been continued to one or two more places, then, after the multiplication, 2 would have been carried to the ſeventh place of decimals, and the product would have agreed with the logarithm found in the Table.

46. *To extract the root of any number.*

Rule. Divide the logarithm of the given number, by the number denoting the root to be extracted, and the quotient is the logarithm of the root (7).

Ex. *What is the 5th root of 784?*

5)2,8943161	log. of - - -	784
0,5788632	——— - -	3,792044 the root.

47. If the given number be leſs than unity, and you expreſs it's logarithm by a negative index, the operation is performed by the following

Rule. Reduce the logarithm of the number to a *negative* logarithm (30), and then divide it by the number denoting the root to be extracted, and you have the logarithm of the root expreſſed by a *negative* logarithm; reduce this back again to a logarithm with a negative index and poſitive decimal part (30), and you have the *true* logarithm of the root.

Ex. 1. *What is the* 0,72 *root of* 0,096?

The logarithm of 0,096 is $\bar{2},9822712 = -1,0177288$, and $\frac{-1,0177288}{0,72} = -1,4135122 = \bar{2},5864878$ the log. of 0,03859116 the root required.

Ex. 2. *What is the* 504 *root of* 0,2?

The log. of 0,2 is $\bar{1},3010300 = -0,6989700$, and $\frac{-0,6989700}{504} = -0,0013868 = \bar{1},9986132$ the logarithm of 0,9968119 the root required

Ex. 3. *What is the* 0,1 *root of* 0,1?

The log. of 0,1 is $\bar{1},0000000$; and as here is no decimal part, we have $\frac{-1}{0,1} = -10 = \overline{10},0000000$ the logarithm of 0,0000000001 the root required.

This Rule for the extraction of roots of numbers leſs than unity, admits of no variety of caſes in practice.

48. *To find the value of a quantity expreſſed by a number whoſe index is a vulgar fraction.*

Rule. Multiply the logarithm of the number by the numerator of the fraction denoting the index, and divide the product by the denominator, and the quotient is the logarithm of the quantity required. But if the given number be a decimal, reduce it's logarithm to a *negative* logarithm (30); multiply it by the numerator of the fraction expreſſing the index, and divide the product by the denominator, and reduce the *negative* logarithm back to the logarithm with a negative index only, and you get the logarithm of the quantity required.

Ex. *What is the value of* $\overline{0,096}\big|^{\frac{5}{9}}$?

The log. of 0,096 is $\bar{2},9822712 = -1,0177288$; hence,

$$\begin{array}{r} -1,0177288 \\ 5 \\ \hline 9)\,-5,0886440 \\ \hline -0,5654049 = \bar{1},4345951 \\ \hline \end{array}$$

the logarithm of 0,2720164 the quantity ſought.

As the numerator of the index denotes the power to which the number is to be raiſed, and the denominator denotes the root of that power to be extracted, the reaſon of the operation is manifeſt from Art. 6. and 7; for by firſt multiplying the logarithm of the number by the numerator of the index, you have the logarithm of the power, and then dividing by the denominator, you have the logarithm of the root of that power.

49. *To find a fourth proportional to three given numbers.*

The fourth term of a proportion is found by multiplying the ſecond and third terms together, and dividing the product by the firſt; the operation therefore is performed by Art. 40. Hence, we have the following

Rule. Let a, b, c, be the firſt, ſecond and third terms; put m = the number of decimal figures in a, n = the number in b and c together; and let A, B, C, be the reſpective values of a, b, c, conſidering the ſignificant figures of each term as whole numbers; then the logarithm of the fourth term = ar. co. log. A + log. B + log. $C + m - n - 10$.

Ex. 1. *What is the fourth proportional to 24, 35, and 79?*

Here, $m = 0$, $n = 0$; hence, we have to ſubtract 10 from the index.

8,6197888	ar. co. log. of - - - -	24
1,5440680	log. of - - - - - -	35
1,8976271	——— - - - - - -	79
2,0614839	——— - - - -	115,2083 the

fourth proportional.

Ex. 2. *What is the fourth proportional to* 0,073, 0,42, *and* 0,009?

Here, $m = 3$, $n = 5$, $m - n - 10 = -12$; hence, we have to ſubtract 12 from the index.

8,1366771	ar. co. log. of - - - -	73
1,6232493	log. of - - - - - - - -	42
0,9542425	——— - - - - - - -	9
$\bar{2}$,7141689	——— - - - -	0,0517808 the

fourth proportional.

Here, the index of the ſum being 10, by ſubtracting 12 from it, the remainder is -2.

50. In trigonometrical operations, when radius is the firſt term, it's logarithm in the Tables is 10,0000000, and therefore it's arithmetic complement is 0,0000000; hence, we have only to add the logarithms of the ſecond and third terms together, and ſubtract 10 from the index. If radius be one of the mean terms, we then add 10 to the logarithm of the other mean term, and ſubtract the logarithm of the firſt term from it. In the Tables of natural ſines, coſines, &c. we generally ſuppoſe radius $=1$, and therefore it's logarithm $=0$; but in theſe Tables the logarithm of radius is 10,0000000; if therefore we make any natural number, the ſine, coſine, &c. of an arc, to radius unity, and we want to expreſs it's logarithm ſo as to agree with the logarithm of the ſines, coſines, &c. in the Table of thoſe quantities, we muſt add 10 to the index of the logarithm, as taken from the Table of the logarithms of the natural numbers. And, *vice verſâ*, if we want to reduce the logarithm of a ſine, coſine, &c. as found in the Table of thoſe quantities, to the logarithm of the natural numbers, in order to determine the value of ſuch a ſine, coſine, &c. we muſt ſubtract 10 from the index.

51. *To find any number of geometrical means between two given numbers.*

RULE. Take the difference between the logarithms of the two numbers, and divide it by the number of means increaſed by unity, and add the quotient continually to the leaſt logarithm, and you get the logarithms of the means.

Ex. *Find four geometrical means between* $10\frac{2}{3}$ *and* $14\frac{7}{9}$.

The logarithm of $10\frac{2}{3}$ = log. of $\frac{32}{3}$ = log. 32 − log. 3 = 1,5051500 − 0,4771213 = 1,0280287; and log. of $14\frac{7}{9}$ = log. of $\frac{133}{9}$ = log. 133 − log. 9 = 2,1238516 − 0,9542425 = 1,1696091. The difference of theſe logarithms is 0,1415804, which divided by 5, the quotient is 0,0283161; hence,

1,0280287
0,0283161
———
1,0563448 log. of 11,38531 – 1ſt mean.
1,0846609 —— 12,15236 – 2d
1,1129770 —— 12,97111 – 3d
1,1412931 —— 13,84500 – 4th

The reaſon of this operation is, that when the numbers are in geometrical progreſſion, their logarithms are in arithmetical progreſſion (3); and the number of intervals of the terms is always greater by unity than the number of means, and therefore you muſt divide the difference of the logarithms by a number greater by unity than the number of means.

52. *Given the hypothenuſe (h) of a right angled plane triangle, and one leg (l), to find the other leg (L).*

By Euc. B. 1. p. 47. $L=\sqrt{h^2-l^2}=\sqrt{\overline{h+l}\times\overline{h-l}}$; hence (4, 7), log. $L=\frac{1}{2}(\log.\,\overline{h+l}+\log.\,\overline{h-l})$.

Ex. *Let* $h=48796$, $l=21435$; then

$$\begin{array}{lcr}
h+l=70230 & \text{- -} & \log.\ \ 4{,}8465227 \\
h-l=27361 & \text{- -} & \log.\ \ \underline{4{,}4371320} \\
 & & 2)\underline{9{,}2836547} \\
L=43835{,}64 & \text{- -} & \log.\ \ \underline{4{,}6418273}
\end{array}$$

The application of logarithms to trigonometrical computations, will be found in the following part of this work.

HYPERBOLIC LOGARITHMS.

53. **If from the center of an equilateral hyperbola,** we take upon one of it's aſymptotes, an abſciſſa which ſhall be equal to the correſponding ordinate, and repreſent each by unity; then if $1+x=$ any other abſciſſa, the correſponding ordinate $=\frac{1}{1+x}$; and (*Prin. of Flux.* **Art. 49. Ex. 3.**) the area comprehended between the

ordinates 1 and $\frac{1}{1+x}$, is the logarithm of $1+x$ to the *modulus* unity. Hence, theſe logarithms are called, *Hyperbolic* logarithms; and a Table thus conſtructed is very uſeful for finding fluents; for the modulus being unity, the fluent found by theſe logarithms requires no multiplication. The general equation expreſſing the relation between a logarithm (x) and it's natural number (b), is $a^x=b$ (9); and in this ſyſtem, $a=2{,}7182818$; hence, the equation is $2{,}7182818^x=b$. If $b=2$, $x=0{,}69314708$ the hyperbolic logarithm of 2. Now the common logarithm of 2 is 0,3010300; and the ratio of 0,69314708 : 0,3010300 is 1 : 0,43424948; therefore 0,43424948 is the modulus of the common ſyſtem; becauſe the logarithms of any number are to each other as the moduli of the ſyſtems (*Fluxions*, Art. 103.) Hence, if we divide the common logarithm by 0,43424948, or multiply it by the reciprocal thereof 2,302585, we get the hyperbolic logarithm.

A Table of hyperbolic logarithms would not be ſo convenient for practice, as that now in uſe; for in the latter ſyſtem, the index of the logarithm is more readily known, it being always leſs by unity than the number of integral figures of the given number, on which account, the computations by this ſyſtem are made more readily, and with greater certainty.

LOGISTIC

LOGISTIC LOGARITHMS.

54. Beſides the common logarithms here treated of, there are others called *Logiſtic* logarithms, which are the common logarithms ſubtracted from 3,5563, the logarithm of 3600, the number of ſeconds in 60′. By this means, the logarithm of 3600″ (=60′) becomes nothing; and the logarithm of 360 is 1,0000. For numbers greater than 3600, the logarithms would be negative, but inſtead of putting them down ſo, their arithmetic complements are put down. Theſe logarithms are frequently uſeful in aſtronomical computations, as it very often happens in a proportion, that the firſt, or one of the mean terms, is 60′.

Rule. If the firſt term be 60′, or 3600″, add the logarithms of the ſecond and third terms together, and the ſum is the logarithm of the fourth term; remembering that if the ſum of the two laſt figures in the addition, the index excepted, be equal to or greater than 10, you are not to carry 1 to the index. If one of the mean terms be 60′, ſubtract the logarithm of the firſt term from the logarithm of the other mean, adding 1 to the index of the logarithm of that mean, if neceſſary, and the remainder is the logarithm of the fourth term.

Ex. 1. *What is the fourth proportional to* 60′, 65′. 25″ *and* 37′. 41″?

65′. 25″ - - - - - log. 9625
37. 41 - - - - - log. 2020

Anſwer 41. 5 - - - - log. 1645

Ex. 2. *What is the fourth proportional to* 60′, 1′. 36″, *and* 27′. 38″?

1′. 36″ - - - - - log. 1,5740
27. 38 - - - - - log. 3367

Anſwer 0. 44 - - - - log. 1,9107

Ex. 3. *What is the fourth proportional to* 16′. 47″, 60′, *and* 17′. 28″?

17′. 28″ - - - - - log. 5359
16. 47 - - - - - log. 5533

Anſwer 62. 27 - - - - log. 9826

Here, it was neceſſary to add 1 to the index of the logarithm of the third term, or which is the ſame, to add 10 to the laſt figure 5 of the upper line, Ex. 4. *What is the fourth proportional to* 27′. 19″, 60′, *and* 5′. 9″?

5′. 9″	log.	1,0663
27. 19	log.	3417
Anſwer 11. 19	log.	7246

If the firſt term be 24 hours, and the ſecond term be hours and minutes, and the third term be given in time, or be an arc, we may find a fourth proportional, by conceiving the head of the Table to repreſent hours.

Ex. *What is the fourth proportional to* 24*h.*, 13*h.* 53′, *and* 76′. 34″?

24*h.*	ar. co. log.	6021
13. 53′	log.	6357
76. 34″	log.	8941
Anſwer 44. 17	log	1319

We here reject 2 in the index, becauſe the firſt and third terms are arithmetic complements, and 1 is to be rejected for each.

In like manner, whatever may be the three terms, whether hours and minutes, minutes and ſeconds of time; degrees and minutes, minutes and ſeconds of an arc; or two of one, and one of the other, a fourth proportional may be found, provided the quantities fall within the limit of the Table.

PROPORTIONAL LOGARITHMS.

55. In the *Requiſite Tables*, there is a Table of *proportional* logarithms, which are analogous to the logiſtic logarithms, the common logarithms being here ſubtracted from 4,0334, the logarithm of 10800, the number of ſeconds in 180′, which is 3 hours, or 3°, the logarithm of which is therefore nothing. Theſe logarithms are peculiarly adapted to the purpoſe of finding the apparent time at Greenwich, by comparing the true diſtance of the moon and ſun, or moon and ſtar, found from the obſerved diſtance, with the true diſtance, put down in the *Nautical Almanac* for every 3 hours, under the meridian of Greenwich.

Ex. 1. *On July* 7, 1775, *the true diſtance of the Sun from the Moon was found to be* 109°. 34′. 26″; *to find the time at Greenwich.*

By the *Nautical Almanac*, the true diſtance is found to be 108°. 5′. 58″ at 3 o'clock, and 109°. 37′. 16″ at 6 o'clock; the moon has therefore receded from the ſun 1°. 31′. 18″ in 3 hours; now the difference between 108°. 5′. 58″ and 109°. 34′. 26″ is 1°. 28′. 28″; to find therefore how long the moon will be in receding 1°. 28′. 28″, we have (upon ſuppoſition that the moon recedes uniformly from the ſun, which is very nearly true) 1°. 31′. 18″ : 1°. 28′. 28″ :: 3*h*. : 2*h*. 54′. 25″, which operation is thus performed by *Proportional Logarithms*, and the whole work may be arranged as follows.

True

True diſt. of ☽ from ☉	109°. 34′. 26″	
——— at 3^h. *(N. Alm.)*	108. 5. 58	
——— at 6^h. ———	109. 37. 16	
Diff. bet. 1ſt and 2d -	1. 28. 28	pr. log. 0,3085
——— 2d and 3d -	1. 31. 18	——— 0,2948
	2. 54. 25	——— 0,0137
	3. 0. 0	
Time at Greenwich -	5. 54. 25	

The third term being here 3 hours, it's logarithm is 0; we have therefore only to ſubtract the log. of the firſt term from that of the ſecond, and the difference 0,0137 is the log. of 2*h*. 54′. 25″, the time in which the moon recedes 1°. 28′. 28″ from the ſun; and as this is the ſpace through which the moon has receded from the ſun after 3 o'clock, if you add 2*h*. 54′. 25″ to 3*h*. you get 5*h*. 54′. 25″, the time at which the moon is at the diſtance 109°. 34′. 26″ from the ſun.

Ex. 2. *What is the fourth proportional to* 2°. 7′. 48″, 1°. 19′. 27″, *and* 3°?

	1°. 19′. 27″ pr. log. - -	0,3552
	2. 7. 48 ——— - -	0,1487
Fourth prop.	1. 51. 53 ——— - -	0,2065

If in any caſe, 3*h*., or 3°, ſhould be the firſt term of the proportion, the logarithm of the fourth term is found by adding together the logarithms of the ſecond and third terms.

Ex. 3. *What is the fourth proportional to* 3°, 2°. 44′. 7″, *and* 1°. 15′. 3″?

	2°.	44′.	7″	pr. log. - -	0,0401
	1,	15.	3	——— - -	0,3800
Fourth prop.	1.	8.	25	——— - -	0,4201

◆

PLANE

PLANE

TRIGONOMETRY.

DEFINITIONS, &c.

Art. 56. TRIGONOMETRY is a part of Geometry which relates to triangles; and treats of the analogy between the ſides and angles, arcs and angles being eſtimated by certain right lines which have relation to them; by which, having given three parts of a triangle, the other three may be found, in all caſes where the data are ſufficient.

57. The circumference of every circle is ſuppoſed to be divided into 360 equal parts, called *Degrees*; every degree into 60 equal parts called *Minutes*; every minute into 60 equal parts called *Seconds*; &c.

58. The characters denoting degrees, minutes and ſeconds, are °, ′, ″; thus, 35°. 17′. 52″ denote 35 degrees, 17 minutes, 52 ſeconds.

59. The angle of a ſector of a circle is in proportion to the arc of that ſector (Euc. B. VI. p. 33.); the arcs therefore of a circle being the meaſures of

the angles at the center, the angles are meaſured by degrees, minutes, and ſeconds of their correſponding arcs.

FIG. 1. 60. Let *C* be the center of the circle *ABDE*, *ACD* a diameter, and *BCE* perpendicular to it; aſſume any arc *AM*; draw *MH*, *AT* perpendicular to *ACD*, and let *AT* meet *CM* produced in *T*; draw alſo *BN*, *MQ* perpendicular to *BC*. Now as all the angles about *C* make 4 right angles (Euc. B. I. p. 15. c. 2.), *AB* is a fourth part of the circumference, or an arc of 90°, and *ABD* is half the circumference, or an arc of 180° (Euc. B. VI. p. 33).

61. The *Complement* of an arc, or of an angle, is what it wants of 90°; the ſame term is alſo ſometimes uſed for the exceſs of an arc above 90°; and the *Supplement* of an arc, or of an angle, is what it wants of 180°. Hence, the arc *MB* is the complement of the arc *MA*, and the angle *MCB* is the complement of the angle *MCA*; alſo, the arc *MD* is the ſupplement of the arc *MA*, and the angle *MCD* is the ſupplement of the angle *MCA*.

62. The *Sine* of an arc, is a line drawn from one extremity of the arc perpendicular to a diameter drawn through the other extremity. Hence, *MH* is the ſine of *MA*, and of *MD*.

63. Cor. Hence, the ſine of an arc is the ſame as the ſine of it's ſupplement.

64. The *Coſine* of an arc, is that part of the diameter which is intercepted between the ſine and the center. Hence, *CH* is the coſine of *AM*.

65. Hence, if the hypothenuſe of a right angled triangle be made radius, the perpendicular will be the ſine, and the baſe the coſine of the angle at the baſe.

66. Cor. As the angle $QCA=MHA$, each being a right angle, QC is parallel to MH (Euc. B. i. p. 28); for the ſame reaſon, MQ is parallel to HC; therefore $HMQC$ is a parallelogram; and $MQ=CH$, $MH=CQ$ (Euc. B. i. p. 34). But MQ is the ſine of MB (62), and CH is the coſine of MA (64); alſo, MH is the ſine of AM (62), and CQ is the coſine of MB (64). Hence, the ſine and coſine of an arc are reſpectively equal to the coſine and ſine of it's complement.

67. The *Verſed-ſine* of an arc, is that part of the diameter which is intercepted between the ſine and the arc. Hence, AH is the verſed-ſine of AM.

68. The *Tangent* of an arc, is a line drawn from one extremity of the arc, perpendicular to the radius*, and terminated by producing the radius drawn to the other extremity. Hence, AT is the tangent of AM.

69. Cor. If the baſe of a right angled triangle be made radius, the perpendicular will be the tangent of the angle at the baſe.

70. The *Secant* of an arc, is the radius produced to meet the tangent. Hence, CT is the ſecant of AM.

71. Cor. If the baſe of a right angled triangle be made radius, the hypothenuſe will be the ſecant of the angle at the baſe.

72. The *Cotangent* of an arc, is the tangent of the complement of that arc. Hence, BN is the cotangent of AM.

73. The *Coſecant* of an arc, is the ſecant of the

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* That a tangent is perpendicular to the radius, ſee Euc. B. iii. p. 16. Cor.

complement of that arc. Hence, CN is the coſecant of AM.

74. As an angle is meaſured by the arc which ſubtends it (59), the terms *ſine*, *coſine*, &c. of an arc are applied to the correſponding angle.

Theſe lines which are related to the arcs of circles, or their correſponding angles, are all that are neceſſary for the various computations in Trigonometry.

PROP. I.

The ſame angle is meaſured by the ſame number of degrees, whatever be the radius of the circle.

FIG. 2. 75. For if O be the center of the circles ABC, abc, and OaA, ObB, be any two radii, then (EUC. B. VI. p. 33.)

$$AB : \text{circum. } ABCA :: \text{ang. } AOB : 4 \text{ right ang.}$$
$$ab : \text{circum. } abca :: \text{ang. } aOb : 4 \text{ right ang.}$$

Hence, AB : circum. $ABCA$ :: ab : circum. $abca$; therefore AB being the ſame part of $ABCA$ that ab is of $abca$, if each circumference be divided into an equal number of degrees, AB, ab, muſt alſo contain the ſame number.

COR. The angle $AOB = \frac{AB \times 4 \text{ right ang.}}{\text{circum. } ABCA}$; and as 4 right angles is a conſtant quantity, and the circumferences of circles vary as their radii, the angle AOB varies as $\frac{AB}{AO}$; that is, *Angles are to each other as their arcs directly and the radii inverſely.*

PROP.

PROP. II.

Given the ſine MH of an arc AM, to find the coſine, verſed-ſine, tangent, ſecant, cotangent, and coſecant. FIG I.

76. By Euc. B. 1. p. 47. $CM^2 = CH^2 + MH^2$; hence, $CH = \sqrt{CM^2 - HM^2}$ the *coſine*.

77. The *verſed-ſine* $AH = AC - CH = \text{rad.} - \text{coſine}$. If H lie on the other ſide of C, then $AH = AC + CH = \text{radius} + \text{coſine}$, conſidering here the coſine to be a poſitive quantity; but if we conſider CH in the former caſe to be poſitive, it will be negative in the latter caſe (ſee the *Algebra*, Art. 473.); conſidering it therefore as a negative quantity in the latter inſtance, we have $AH = AC - CH$ as in the former caſe, for $-$ before a negative quantity makes it additive.

78. By ſimilar triangles, CHM, CAT, $CH : HM :: CA : AT = \dfrac{CA \times HM}{CH} = \dfrac{\text{rad.} \times \text{ſin.}}{\text{coſ.}}$ the *tangent*.

79. COR. When $HM = CH$, the angles HCM, HMC are equal, and therefore $HCM = 45°$, and the tangent $AT = CA$ the radius.

80. By the ſame triangles, $CH : CM$, or CA, $:: CA : CT = \dfrac{CA^2}{CH} = \dfrac{\text{rad.}^2}{\text{coſ.}}$ the *ſecant*. Or, $MH : CM :: AT : CT = \dfrac{CM \times AT}{MH} = \dfrac{\text{rad.} \times \text{tan.}}{\text{ſin.}}$ the *ſecant*. Alſo, the *ſecant* $= \sqrt{CA^2 + AT^2} = \sqrt{\text{rad.}^2 + \text{tan.}^2}$

81. As the triangles CHM, CBN, have the angles at H and B right ones, and the angle $= HMC =$ alt. angle BCN, theſe triangles are ſimilar, and $HM : CH :: CB : BN = \dfrac{CB \times CH}{HM} = \dfrac{\text{rad.} \times \text{coſ.}}{\text{ſin.}}$ the *cotangent*.

82. Hence (78, 81), the cotangent = rad.² divided by the tangent; conſequently, tangent : rad. :: rad. : cotangent.

83. By the ſame triangles, $HM : CM$, or CB, $:: CB : CN = \frac{CB^2}{HM} = \frac{\text{rad.}^2}{\text{ſin.}}$ the *coſecant*. Or, $CH : CM :: BN : CN = \frac{CM \times BN}{CH} = \frac{\text{rad.} \times \text{cot.}}{\text{coſ.}}$ the *coſecant*.

84. Cor. 1. In this propoſition we have proved the following proportions:

Sine : radius :: radius : coſecant.
Coſine : radius :: radius : ſecant.
Tangent : radius :: radius : cotangent.

Hence, as in the Tables, log. radius = 10,0000000, we have, log. ſine + log. coſecant = 2 log. radius = 20,0000000, or, log. ſine + log. coſecant − 10, = 10,0000000; therefore (29) the arithmetic complement of log. ſine is log. coſecant − 10,. For the ſame reaſon, the arithmetic complement of log. coſine = log. ſecant − 10,. Alſo, the arithmetic complement of log. tan. = log. cotan. − 10,. But if the tangent be greater than radius, and it be required to take it's arithmetic complement in order to convert a ſubtractive quantity into one that is additive, then you muſt take the log. cotangent, and after the addition, reject 20 from the index, inſtead of 10 in the other caſes. Thus you may immediately take the arithmetic complements of the quantities required. In Taylor's Logarithms there are no ſecants and coſecants, and therefore in thoſe Tables, the arithmetic complements of the ſines and coſines muſt be taken as directed in Art. 29.

85. Cor. 2. Let us ſuppoſe the arc to begin at *A*, and in the quadrant *AB*, aſſume the ſine, coſine, tangent, ſecant, cotangent, and coſecant to be poſitive. Now if a line be poſitive when ſet off in one direction, if it vaniſh, and then be ſet off in an oppoſite direction, it becomes negative (ſee the *Algebra*, Art. 473). Hence, for the arc *Am*, or *Am′*, the coſine *Ch* is negative; but for the arc *AM′*, the coſine *CH* is poſitive. For the arc *ABD*, the ſine is ſet off in the ſame direction, and is therefore poſitive; but at *D* it paſſes through 0, and for the arc *DEA* it is ſet off in an oppoſite direction, and is therefore negative. The tangent = radius × ſine ÷ coſine (78); now radius is poſitive; and from *A* to *B*, the ſine, coſine and tangent are poſitive by ſuppoſition. At *B*, the coſine = 0; but there is no tangent nor ſecant at that point, as *CB* produced never meets *AT*, they being parallel. An arc of 90° therefore has not (according to the definition of the terms) either a tangent or ſecant *. From *B* to *D*, the ſine is poſitive, and the

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* Difficulties have frequently ariſen, in conſequence of it's being ſuppoſed, that an arc of 90° has a tangent and ſecant, each infinite. For inſtance, in a right angled ſpherical triangle, radius : coſine of the angle at the baſe :: tangent of the hypothenuſe : tangent of the baſe; now when the baſe = 90°, the hypothenuſe = 90°; and therefore theſe arcs being equal, if they have any tangents, of whatever value they may be, they muſt be equal, and therefore radius = coſine of the angle at the baſe, whatever that angle may be. This falſe concluſion ariſes from the ſuppoſition, that an arc of 90° has a tangent. As the arc increaſes till it becomes 90°, the tangent and ſecant increaſe without limit, and at 90° the arc ceaſes to have either a tangent or ſecant. As the arc by increaſing paſſes through 90°, the tangent and ſecant increaſe without limit, ceaſe to exiſt at 90°, and then begin again at a quantity

coſine is negative; therefore the tangent is negative. From D to E, the ſine and coſine are negative; therefore the tangent is poſitive. From E to A, the ſine is negative, and the coſine is poſitive; therefore the tangent is negative. The ſecant $= \text{rad.}^2 \div$ coſine; therefore the ſecant has the ſame ſign as the coſine. The cotangent $= \text{rad.}^2 \div$ tangent; therefore the cotangent has the ſame ſign as the tangent. The coſecant $= \text{rad.}^2 \div$ ſine; therefore the coſecant has the ſame ſign as the ſine. The verſed-ſine continues poſitive through the whole circle, it being always ſet off from A in the ſame direction. Hence,

		From 0° to 90°	From 90° to 180°	From 180° to 270°	From 270° to 360°
Sine	is	+	+	−	−
Coſine	is	+	−	−	+
Tangent	is	+	−	+	−
Secant	is	+	−	−	+
Cotangent	is	+	−	+	−
Coſecant	is	+	+	−	−
Verſed-ſine	is	+	+	+	+

86. If we conſider the arc AM as poſitive, the arc AM', ſet off from A in a contrary direction, will be negative; and it will be uſeful to conſider, how the ſigns of the ſine, coſine, verſed-ſine, tangent, cotangent, ſecant, and coſecant will be affected in the latter caſe, compared with the ſigns of the ſame quantities in the former. The arcs AM, AM' are equal, and have the ſame coſine CH, and ſame verſed-ſine AH; [illegible] but [illegible]

indefinitely great. And thus in other caſes where the tangent or ſecant of an arc enter into the computation, when the arc becomes 90° we can draw no concluſion on which we can depend.

but the ſine *HM'* being ſet off in a direction contrary to that of the ſine *HM*, the ſine of the negative arc *AM'* becomes negative, or contrary to the ſign of the ſine of a poſitive arc *AM* of the ſame value. Now as tan. = rad. × ſin. ÷ coſ. (78), cotan. = rad. × coſ. ÷ ſin. (81), coſec. = $\text{rad.}^2 \div \text{ſin.}$ (83), theſe quantities will have a different ſign for a negative arc; but as ſec. = $\text{rad.}^2 \div \text{coſ.}$ (80), the ſecant will have the ſame ſign for a negative as for a poſitive arc. If therefore we have the ſine, tangent, cotangent, or coſecant of $A - B$, when *A* is leſs than *B*, the ſigns of it's ſine, tangent, cotangent, and coſecant, will be contrary to thoſe which are given in the preceding Table.

87. In trigonometrical computations, when the quantity required comes out in terms of the ſine, ſince any angle or arc, and it's ſupplement, have the ſame ſine (63), the caſe is ambiguous, unleſs the ambiguity be taken away by ſome other conſideration; but if the quantity ſought be expreſſed by a coſine, or tangent, there will be no ambiguity, as a poſitive coſine or tangent denotes an arc or angle leſs than 90°, and a negative coſine or tangent denotes an arc or angle between 90° and 180°; and in Trigonometry, every angle or arc muſt be leſs than 180°.

88. In all trigonometrical calculations, it is neceſſary to attend to the ſigns of the quantities. And to facilitate ſuch computations, Tables are conſtructed to a radius = 1, or = 10000, ſhowing the lengths of the ſines, coſines, tangents, &c. to every degree and minute of the quadrant. Theſe Tables contain alſo the logarithms of the ſame quantities, making 10,0000000 the logarithm of radius, which anſwers to 10000000000 in the common Tables of logarithms.

89. As the arc increaſes from 90° to 180°, the ſine, tangent, and ſecant decreaſe; for by increaſing Am, you decreaſe mh, Dt, Ct. Of two arcs therefore between 90° and 180°, the greater arc has the leſs ſine, tangent, and ſecant.

Prop. III.

The ſine of an angle is equal to half the chord of twice that angle.

90. For produce MH to meet the circumference at M', and join CM'; then $MH = HM'$ (Euc. B. iii. p. 3.), $CM = CM'$, and CH is common to the triangles CMH, $CM'H$; hence, the angle $MCH = M'CH$ (Euc. B. i. p. 8.), or the angle MCM' is twice MCH, and MH, the ſine of the latter, is equal to half MM' the chord of the former angle.

91. Cor. Let the angle $MCH = 30°$, then the angle $MCM' = 60°$; and as $CM = CM'$, the angle $CMM' = CM'M$, and therefore each of theſe angles $= 60°$; conſequently CMM' is an equilateral triangle, and $MM' = MC$; that is, the chord of 60° = radius; hence, the ſine of 30° = ½ radius; therefore the coſine of 60° = ½ radius (66); conſequently the verſed-ſine of 60° (= radius − coſine) = ½ radius.

Prop. IV.

Radius : the ſine of an angle :: twice the ſine of that angle : the verſed-ſine of twice the ſame angle.

FIG. 3. 92. Let C be the center of the circle AMB, ACM any angle; join MA, and draw CDE perpendicular to it; then $AD = DM$ (Euc. B. iii. p. 3.), and the angle *ACD*=*MCD* (Euc. B. I. p. 8.); draw alſo *MH* perpendicular to *ACB*. Now the triangles *CDA*, *MHA* are ſimilar, for they are right angled at *D* and *H*, and the angle at *A* is common; hence, *CA* : *AD* :: *AM*, or 2*AD*, : *AH*, where *AD* is the ſine of *ACD*, and *AH* is the verſed-ſine of *ACM*, or 2*ACD*.

93. Cor. Put r=radius; then as $AH=\frac{2AD^2}{CA}$, $CH=CA-AH=CA-\frac{2AD^2}{CA}$; that is, $\text{coſ.}\ ACM=r-\frac{2\times\overline{\text{ſin.}\ \frac{1}{2}ACM}^2}{r}$; hence, $r-\text{coſ.}\ ACM=\frac{2\times\overline{\text{ſin.}\ \frac{1}{2}ACM}^2}{r}$.

Prop. V.

If A repreſent an angle, then $\frac{r-\text{coſ.}\ A}{r+\text{coſ.}\ A}=\frac{\overline{\text{tan.}\ \frac{1}{2}A}^2}{r^2}$.

94. Join *MB*, and draw *CL* perpendicular to it; then *BL*=*LM* (Euc. B. III. p. 3.); and as *BLC* is a right angle, it is equal to *BMA* (Euc. B. III. p. 31.), therefore *CL* is parallel to *DM* (Euc. B. I. p. 28.); for the ſame reaſon, *CD* is parallel to *LM*; therefore *CDML* is a parallelogram, and *CD*=*LM*=*BL*. Now the triangles *BLC*, *BMH* are ſimilar, for they are right angled at *L* and *H*, and the angle at *B* is common; hence, *BC* : *BL* :: *BM*, or 2*BL*, : *BH* $=\frac{2BL^2}{BC}$, or $BC+CH=\frac{2CD^2}{BC}$; that is, $r+\text{coſ.}\ ACM$ =

$= \frac{2 \times \text{coſ.} \frac{1}{2} \overline{ACM}^2}{r}$; hence, from this and the laſt Article, we get $\frac{r - \text{coſ.} ACM}{r + \text{coſ.} ACM} = \frac{\overline{\text{ſin.} \frac{1}{2} ACM}^2}{\overline{\text{coſ.} \frac{1}{2} ACM}^2} = \frac{\overline{\text{tan.} \frac{1}{2} ACM}^2}{r^2}$ (78).

Prop. VI.

The verſed-ſine of an angle multiplied into half the radius, is equal to the ſquare of the coſine of half the ſupplement of that angle.

95. For $CD = \frac{1}{2} BM$, therefore $CD^2 = \frac{1}{4} BM^2 =$ (Euc. B. vi. p. 8.) $\frac{1}{4} BA \times BH = \frac{1}{2} AC \times BH$; in which equation, BH is the verſed-ſine of BM, and CD is the coſine of AE, or $\frac{1}{2} AM$, or half the ſupplement of BM.

Prop. VII.

Radius : the coſine of an angle :: twice the ſine of that angle : the ſine of twice the ſame angle.

96. For from the ſimilar triangles CAD, MAH, $CA : CD :: AM$, or $2AD$, $: MH$, where CD is the coſine, and AD the ſine of AE, and MH is the ſine of AM, or $2AE$.

97. Cor. Hence, if A be any angle, ſin. $A \times$ coſ. $A = \frac{1}{2}$ rad. $\times$ ſin. $2A$.

Prop.

PROP. VIII.

The tangent + cotangent of an arc = 2 coſecant of twice that arc.

98. Let t = tangent of an arc A; then (82) it's cotangent $= \frac{r^2}{t}$; and $t + \frac{r^2}{t} = \frac{t^2+r^2}{t} = \frac{\overline{\text{ſec.}}|^2}{t}$ (80) $= \left(\text{as } \frac{\text{ſec.}}{t} = \frac{r}{\text{ſin.}} \text{ (80)}\right) \frac{r \times \text{ſec.}}{\text{ſin.}} = \left(\text{as ſec.} = \frac{r^2}{\text{coſ.}} \text{ (80)}\right) \frac{r^3}{\text{ſin.} \times \text{coſ.}} = \left(\text{as ſin.} \times \text{coſ.} = \tfrac{1}{2} r \times \text{ſin. } 2A \text{ (97)}\right) \frac{2r^2}{\text{ſin. } 2A} = \left(\text{as } \frac{r^2}{\text{ſin. } 2A} = \text{coſec. } 2A \text{ (83)}\right)$ 2 coſec. $2A$.

PROP. IX.

Given the ſines and coſines of two angles, to find the ſines and coſines of their ſum *and* difference.

99. Let ACM, MCN be the two given angles, of which ACM is the greater; make the angle $MCP = MCN$, and then ACN is the ſum of the two given angles, and ACP their difference. Draw the chord PN, interſecting CM in D; then as the angle $PCD = NCD$, $PC = CN$, and CD is common to the triangles PCD, NCD, we have (EUC. B. 1. p. 4.) $PD = DN$, and the angle $PDC = NDC$, therefore each of them is a right angle. Draw NG, DB, MH, PK, perpendicular to AC, and therefore parallel to each other, and DF, PxE parallel to AC, and therefore FIG. 4.

parallel to each other (Euc. B. I. p. 30.). Now as $ND = DP$, we have (Euc. B. VI. p. 2.) $NF = FE = Dx$, and $Px = Ex = DF$; therefore $BK = GB = DF$. Alſo, the triangle NDF is ſimilar to the triangle CBD, for the angle $NDF = CDB$, each being the complement of FDC, and the angles at F and B are right angles; but CDB is ſimilar to CMH, the angle at C being common, and the angles at B and H right angles; therefore NDF is alſo ſimilar to CMH. Hence,

$$CM : MH :: CD : DB = \frac{MH \times CD}{CM},$$

$$CM : CH :: ND : NF = \frac{CH \times ND}{CM};$$

Therefore $\frac{MH \times CD + CH \times ND}{CM} = DB + NF = FG + NF = NG$ the *ſine* of the *ſum*. And $\frac{MH \times CD - CH \times ND}{CM} = DB - NF = DB - Dx = Bx = PK$ the *ſine* of the *difference*.

Again,

$$CM : CH :: CD : CB = \frac{CH \times CD}{CM},$$

$$CM : MH :: ND : DF = \frac{MH \times ND}{CM};$$

Therefore $\frac{CH \times CD - MH \times ND}{CM} = CB - DF = CB - BG = CG$ the *coſine* of the *ſum*. And

CH

$\frac{CH\times CD+MH\times ND}{CM}=CB+DF=CB+BK=CK$ the *coſine* of the *difference*.

Hence, if $A=$ the angle ACM, $B=$ the angle MCN, $r=$ radius, we have,

100. $\text{Sin. }(A+B)=\frac{\text{ſin. }A\times\text{coſ. }B+\text{coſ. }A\times\text{ſin. }B}{r}.$

101. $\text{Sin. }(A-B)=\frac{\text{ſin. }A\times\text{coſ. }B-\text{coſ. }A\times\text{ſin. }B}{r}.$

102. $\text{Coſ. }(A+B)=\frac{\text{coſ. }A\times\text{coſ. }B-\text{ſin. }A\times\text{ſin. }B}{r}.$

103. $\text{Coſ. }(A-B)=\frac{\text{coſ. }A\times\text{coſ. }B+\text{ſin. }A\times\text{ſin. }B}{r}.$

Cor. 1. By adding and ſubtracting the equations in Articles 100, 101, and dividing by 2, we get,

104. $\frac{\text{Sin. }A\times\text{coſ. }B}{r}=\tfrac{1}{2}\text{ ſin. }(A+B)+\tfrac{1}{2}\text{ ſin. }(A-B).$

If $A=30°$, $\text{ſin. }A=\tfrac{1}{2}r$ (91); hence, $\text{coſ. }B=\text{ſin. }(30°+B)+\text{ſin. }(30°-B)$; therefore, $\text{ſin. }(30°+B)=\text{coſ. }B-\text{ſin. }(30°-B)$.

105. $\frac{\text{Coſ. }A\times\text{ſin. }B}{r}=\tfrac{1}{2}\text{ ſin. }(A+B)-\tfrac{1}{2}\text{ ſin. }(A-B).$

If $A=60°$, $\text{coſ. }A=\tfrac{1}{2}r$ (91); hence, $\text{ſin. }B=\text{ſin. }(60°+B)-\text{ſin. }(60°-B)$; therefore, $\text{ſin. }(60°+B)=\text{ſin. }(60°-B)+\text{ſin. }B$.

By adding and ſubtracting the equations in Articles 102, 103, and dividing by 2, we get,

106. $\frac{\text{Coſ. }A\times\text{coſ. }B}{r}=\tfrac{1}{2}\text{ coſ. }(A-B)+\tfrac{1}{2}\text{ coſ. }(A+B).$

107. $\frac{\text{Sin.}\,A\times\text{ſin.}\,B}{r}=\frac{1}{2}\,\text{coſ.}\,(A-B)-\frac{1}{2}\,\text{coſ.}\,(A+B).$

Cor. 2. Put $A+B=P$, $A-B=Q$, then $A=\frac{1}{2}(P+Q)$, $B=\frac{1}{2}(P-Q)$; for A and B ſubſtitute theſe values in Articles 104, 105, 106, 107, and multiply by 2, and we get,

108. $\frac{2\,\text{ſin.}\,\frac{1}{2}(P+Q)\times\text{coſ.}\,\frac{1}{2}(P-Q)}{r}=\text{ſin.}\,P+\text{ſin}\;Q.$

109. $\frac{2\,\text{coſ.}\,\frac{1}{2}(P+Q)\times\text{ſin.}\,\frac{1}{2}(P-Q)}{r}=\text{ſin.}\,P-\text{ſin.}\,Q.$

110. $\frac{2\,\text{coſ.}\,\frac{1}{2}(P+Q)\times\text{coſ.}\,\frac{1}{2}(P-Q)}{r}=\text{coſ.}\,Q+\text{coſ.}\,P.$

111. $\frac{2\,\text{ſin.}\,\frac{1}{2}(P+Q)\times\text{ſin.}\,\frac{1}{2}(P-Q)}{r}=\text{coſ.}\,Q-\text{coſ.}\,P.$

112. Cor. 3. From Articles 108, 109, we have, $\frac{\text{ſin.}\,P+\text{ſin.}\,Q}{\text{ſin.}\,P-\text{ſin.}\,Q}=\frac{\text{ſin.}\,\frac{1}{2}(P+Q)\times\text{coſ.}\,\frac{1}{2}(P-Q)}{\text{coſ.}\,\frac{1}{2}(P+Q)\times\text{ſin.}\,\frac{1}{2}(P-Q)}=\frac{\text{tan.}\,\frac{1}{2}(P+Q)}{\text{tan.}\,\frac{1}{2}(P-Q)}$ (78); hence, $\text{ſin.}\,P+\text{ſin.}\,Q:\text{ſin.}\,P-\text{ſin.}\,Q::\text{tan.}\,\frac{1}{2}(P+Q):\text{tan.}\,\frac{1}{2}(P-Q)$. If $P=90°$, and for Q we put $2x$, then $\text{rad.}+\text{ſin.}\,2x:\text{rad.}-\text{ſin.}\,2x::\text{tan.}\,(45°+x):\text{tan.}\,(45°-x)$.

Alſo, from Art. 110, 111, we have $\frac{\text{coſ.}\,Q+\text{coſ.}\,P}{\text{coſ.}\,Q-\text{coſ.}\,P}=\frac{\text{coſ.}\,\frac{1}{2}(P+Q)\times\text{coſ.}\,\frac{1}{2}(P-Q)}{\text{ſin.}\,\frac{1}{2}(P+Q)\times\text{ſin.}\,\frac{1}{2}(P-Q)}=\frac{\text{cot.}\,\frac{1}{2}(P+Q)}{\text{tan.}\,\frac{1}{2}(P-Q)}$ (78, 81); hence, $\text{coſ.}\,Q+\text{coſ.}\,P:\text{coſ.}\,Q-\text{coſ.}\,P::\text{cot.}\,\frac{1}{2}(P+Q):\text{tan.}\,\frac{1}{2}(P-Q)$.

113. COR. 4. From Articles 100, 101, we have ſin. $(A+B)$: ſin. $(A-B)$ :: ſin. $A\times$ coſ. $B+$ coſ. $A\times$ ſin. B : ſin. $A\times$ coſ. $B-$ coſ. $A\times$ ſin. B :: (dividing the two laſt terms by coſ. $A\times$ coſ. B) $\frac{\text{ſin. }A}{\text{coſ. }A}+\frac{\text{ſin. }B}{\text{coſ. }B} : \frac{\text{ſin. }A}{\text{coſ. }A}-\frac{\text{ſin. }B}{\text{coſ. }B}$ :: (78) tan. $A+$ tan. B : tan. $A-$ tan. B :: (82) $\frac{r^2}{\text{cot. }A}+\frac{r^2}{\text{cot. }B} : \frac{r^2}{\text{cot. }A}-\frac{r^2}{\text{cot. }B}$ :: cot. $B+$ cot. A : cot. $B-$ cot. A.

From Articles 102, 103, we have coſ. $(A+B)$: coſ. $(A-B)$ :: coſ. $A\times$ coſ. $B-$ ſin. $A\times$ ſin. B : coſ. $A\times$ coſ. $B+$ ſin. $A\times$ ſin. B :: (dividing the two laſt terms by ſin. $A\times$ coſ. B) $\frac{\text{coſ. }A}{\text{ſin. }A}-\frac{\text{ſin. }B}{\text{coſ. }B} : \frac{\text{coſ. }A}{\text{ſin. }A}+\frac{\text{ſin. }B}{\text{coſ. }B}$ :: (81, 78) cot. $A-$ tan. B : cot. $A+$ tan. B :: (82) $\frac{r^2}{\text{tan. }A}-\frac{r^2}{\text{cot. }B} : \frac{r^2}{\text{tan. }A}+\frac{r^2}{\text{cot. }B}$ :: cot. $B-$ tan. A : cot. $B+$ tan. A.

114. COR. 5. By Art. 78. tan. $A\pm$ tan. $B=r\times\left(\frac{\text{ſin. }A}{\text{coſ. }A}\pm\frac{\text{ſin. }B}{\text{coſ. }B}\right)=r\times\frac{\text{ſin. }A\times\text{coſ. }B\pm\text{ſin. }B\times\text{coſ. }A}{\text{coſ. }A\times\text{coſ. }B}$ $=$(100, 101) $r^2\times\frac{\text{ſin. }(A\pm B)}{\text{coſ. }A\times\text{coſ. }B}$. Alſo (81), cot. $A\pm$ cot. $B=r\times\left(\frac{\text{coſ. }A}{\text{ſin. }A}\pm\frac{\text{coſ. }B}{\text{ſin. }B}\right)=r\times\frac{\text{ſin. }B\times\text{coſ. }A\pm\text{ſin. }A\times\text{coſ. }B}{\text{ſin. }A\times\text{ſin. }B}$ $=$ (100, 101) $r^2\times\frac{\text{ſin. }(B\pm A)}{\text{ſin. }A\times\text{ſin. }B}$.

115. Cor. 6. Let $B=A$, and we have (100) $\text{ſin.}\,2A=\frac{2\,\text{ſin.}\,A\times\text{coſ.}\,A}{r}$, as in Art. 97. Alſo (102), $\text{coſ.}\,2A=\frac{\overline{\text{coſ.}\,A}^2-\overline{\text{ſin.}\,A}^2}{r}=\frac{2\,\overline{\text{coſ.}\,A}^2-\overline{\text{ſin.}\,A}^2-\overline{\text{coſ.}\,A}^2}{r}$ $=\frac{2.\overline{\text{coſ.}\,A}^2-r^2}{r}$.

116. Cor. 7. By Article 78, the tangent of $(A+B)=r\times\frac{\text{ſin.}\,(A+B)}{\text{coſ.}\,(A+B)}=$ (100, 102) $r\times\frac{\text{ſin.}\,A\times\text{coſ.}\,B+\text{coſ.}\,A\times\text{ſin.}\,B}{\text{coſ.}\,A\times\text{coſ.}\,B-\text{ſin.}\,A\times\text{ſin.}\,B}=$ (by dividing the numerator and denominator by coſ. $A\times$ coſ. B, and ſubſtituting tan. ÷ rad. for ſin. ÷ coſ.) $r^2\times\frac{\text{tan.}\,A+\text{tan.}\,B}{r^2-\text{tan.}\,A\times\text{tan.}\,B}$. In like manner, from Art. 101, 103, we get $\text{tan.}\,(A-B)=r^2\times\frac{\text{tan.}\,A-\text{tan.}\,B}{r^2+\text{tan.}\,A\times\text{tan.}\,B}$. But we ſhall ſhow how theſe may be deduced, without firſt finding the ſine and coſine of the ſum and difference.

Prop. X.

Given the tangents of two angles, to find the tangents of their ſum *and* difference.

FIG. 5. 117. Let AOB, AOC be two angles, AG, AT their tangents, and CF the tangent of BOC; draw GD perpendicular to OT, and therefore parallel to FC, and the triangles ODG, OCF are ſimilar, the angle at O being common, and the angles at C and D right ones,

ones. Now the triangles OTA, GTD are ſimilar, being right angled at A and D, and having the angle T common; hence, $OT : AT :: TG$, or $AT-AG$, $: TD = \frac{AT^2-AT\times AG}{OT}$; therefore, $OD = OT - TD = OT - \frac{AT^2-AT\times AG}{OT} =$ (as $OT^2 - AT^2 = OA^2$) $\frac{OA^2+AT\times AG}{OT}$. Alſo, $OT : OA :: TG$, or $AT-AG$, $: DG = OA\times\frac{AT-AG}{OT}$. But by ſimilar triangles, ODG, OCF, $OD : DG :: OC : CF$; that is, $\frac{OA^2+AT\times AG}{OT} : OA\times\frac{AT-AG}{OT} :: OA : CF = OA^2\times\frac{AT-AG}{OA^2+AT\times AG}$ the tangent of the angle BOC, or of the *difference* of the angles AOC; AOB.

Again, from the laſt equation, we get $AT = OA^2 \times\frac{AG+CF}{OA^2-AG\times CF}$ the tangent of the angle AOC, or of the *ſum* of the angles AOB, BOC.

Hence, if T and t be the tangents of two angles A, B, of which T is the greater, then the tangent of $(A+B) = r^2\times\frac{T+t}{r^2-Tt}$, and the tangent of $(A-B) = r^2\times\frac{T-t}{r^2+Tt}$.

118. COR. 1. If $A=B$, $T=t$, and $r^2\times\frac{2T}{r^2-T^2}$ = the tangent of $2A$.

119. Cor. 2. If C, c repreſent the cotangents of the arcs A, B; then as $T=\frac{r^2}{C}$, and $t=\frac{r^2}{c}$ (82), we have, tan. $(A+B)=r^2\times\frac{c+C}{Cc-r^2}$, and tan. $(A-B)=r^2\times\frac{c-C}{Cc+r^2}$. Alſo, tan. $2A=r^2\times\frac{2C}{C^2-r^2}$.

Prop. XI.

Given t the tangent of an arc A, whoſe radius is r, to find the ſine and coſine of twice that arc.

120. Put $n=\sqrt{r^2+t^2}$ the ſecant (80); then $\frac{r^2}{n}=$ coſ. A (80), and $\frac{r^2}{t}=$ cot. A (82); hence (83), $\frac{r^2}{n}:r::\frac{r^2}{t}:\frac{rn}{t}=$ coſec. A; therefore (83) $\frac{rt}{n}=$ ſin. A; hence (96), $r:\frac{r^2}{n}::\frac{2rt}{n}:$ *ſine* of $2A=\frac{2r^2t}{n^2}=\frac{2r^2t}{r^2+t^2}$. Alſo (115), $\frac{r^3-rt^2}{n^2}=\frac{r^3-rt^2}{r^2+t^2}=$ *coſine* of $2A$.

121. Cor. 1. Hence (80), the *ſecant* of $2A=\frac{r^2}{\text{coſ.}}=r^2\times\frac{r^2+t^2}{r^3-rt^2}=r\times\frac{r^2+t^2}{r^2-t^2}$; and (83) the *coſecant* of $2A=\frac{r^2}{\text{ſin.}}=r^2\times\frac{r^2+t^2}{2r^2t}=\frac{r^2+t^2}{2t}$. Now when $2A$ is between 90° and 180°, the expreſſion for it's coſine

becomes negative; therefore (77) the *verſed-ſine* of $2A = r - \frac{r^3 - rt^2}{r^2 + t^2} = \frac{2rt^2}{r^2 + t^2}$.

122. Cor. 2. Alſo (96), the *ſine* of $4A = \frac{4r^4t - 4r^2t^3}{\overline{r^2 + t^2}^2}$; and (115) the *coſine* of $4A = \frac{r^5 - 6r^3t^2 + rt^4}{\overline{r^2 + t^2}^2}$. Now when $4A$ is between 90° and 180°, the expreſſion for it's coſine becomes negative; therefore (77) the *verſed-ſine* of $4A = r - \frac{r^5 - 6r^3t^2 + rt^4}{\overline{r^2 + t^2}^2}$ $= \frac{8r^3t^2}{\overline{r^2 + t^2}^2}$.

Theſe theorems will frequently be found uſeful in rendering equations free from ſurd quantities.

Prop. XII.

In a right angled triangle, the baſe : the perpendicular :: radius : tangent of the angle at the baſe.

123. Let ABC be a triangle, right angled at B; and let An be the radius in the Tables which contain the values of the ſines, coſines, &c. and draw nm perpendicular to An, then nm is the tangent of the angle A (69); and by ſimilar triangles Anm, ABC, AB : BC :: An : nm :: radius : tangent of the angle A. FIG. 6.

Thus, if we have the two legs * of a right angled triangle given, we can find the angles, for having found the angle A, it's complement is the angle C.

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* The ſides including the right angle, are called *Legs.*

124 Cor. If we conſider BC as the baſe, we have, for the ſame reaſon, $BC : BA ::$ radius : tangent of the angle C; and having found one of the angles, from knowing the two legs, the hypothenuſe may be found by Prop. 15. Or it may be otherwiſe found by Prop. 14.

Prop. XIII.

In a right angled triangle, the hypothenuſe : the baſe :: radius : the coſine of the angle at the baſe. Alſo, the hypothenuſe : the perpendicular :: radius : the ſine of the angle at the baſe.

125. For if Am be made radius, An is the coſine, and nm is the ſine of the angle A (65); and by ſimilar triangles, $AC : AB :: Am : An ::$ radius : coſine of the angle A. Alſo, $AC : CB :: Am : mn ::$ radius : ſine of the angle A.

Thus, if we have the hypothenuſe and one leg given, we can find the angles.

The ſolution of the other caſe of right angled triangles, that is, having given one of the oblique angles, and one ſide, to find the other parts, follow immediately from the next propoſition; for if one of the oblique angles be given, the other will be known, the latter being the complement of the former; therefore we ſhall know all the angles and one ſide.

If the hypothenuſe and one leg be given, the other leg may be found by Art. 50. whence the angles are found by Prop. 12. Or, without firſt finding the other leg, the angles may be found by Prop. 15.

PROP. XIV.

Given two legs AB, BC of a right angled triangle ABC, to find the hypothenuſe AC.

126. Let $a=AB$, $b=BC$, then (EUC. B. 1. p. 47.) we have hyp. $AC=\sqrt{a^2+b^2}=a\times\sqrt{1+\frac{b^2}{a^2}}$. Let $An=$ rad. of the Tables $=1$, and draw mn perpendicular to AB; then, by ſimilar triangles, $a:b::1:\frac{b}{a}$ $=mn$ the tangent of the angle A (69); and $Am=\sqrt{1+\frac{b^2}{a^2}}=\text{ſec.}\ A=\frac{\text{rad.}^2}{\text{coſ.}\ A}$ (80). Now as tan. $A=$ rad. of the Tables $\times\ \frac{b}{a}$, and the log. of rad. $=10$, we have (50) log. tan. $A=10$, (log. rad.) + log. b − log. a; and log. $\sqrt{1+\frac{b^2}{a^2}}=20$, − log. coſ. A (36); but to reduce the log. found in the Table of ſines, coſines, &c. to the log. of the natural numbers, you muſt ſubtract 10 from the index (50); find therefore 10, − log. coſ. A, or, which is the ſame, take the arithmetic complement of log. coſ. A, and add it to log. a, and you get the log. of $a\times\sqrt{1+\frac{b^2}{a^2}}$, or the log. of the hypothenuſe.

Ex. *Let $AB = 189$, $BC = 347$; to find AC.*

10, + log. 347 - - - - - 12,5403295
log. 189 - - - - - 2,2764618

log. tan. A - - - - 10,2638677

ar. co. log. coſ. A - - - - 0,3202801
log. 189 - - - - 2,2764618

log. $395{,}13 = AC$ - - 2,5967419

127. Or, AC may be found in the following manner. By Art. 123. $AB : BC ::$ rad. : tan. A; and ſin. A : rad. :: $BC : AC$ (125).

Prop. XV.

In any triangle, the ſides have the ſame proportion to each other, as the ſines of their oppoſite angles.

FIG. 14. 128. Let ABC be any triangle, and draw CD perpendicular to AB; then (125),

$$\begin{aligned} AC : CD &:: \text{rad.} : \text{ſin. } A \\ CD : BC &:: \text{ſin. } B : \text{rad.} \\ \therefore AC : BC &:: \text{ſin. } B : \text{ſin. } A \end{aligned}$$

129. Cor. 1. Hence, if two angles and a ſide oppoſite to one of them be given, the other parts may be found. For let the angles A and ACB be given, then

the angle B will be known, it being the ſupplement of $A+ACB$; let alſo BC be given; then ſin. A : ſin. ACB :: BC : AB; and ſin. A : ſin. B :: BC : AC.

130. Cor. 2. Let two ſides and an angle oppoſite to one of them be given; for inſtance, let the angle A, the ſide AC, and the ſide oppoſite to A be given; and let PQ be the length of the ſide oppoſite to A. *Firſt*, let the angle A be *leſs* than a right angle. With the center C, and radius $=PQ$, deſcribe the circle vw cutting the indefinite line Am, forming the given angle A with AC, in B and E, and join CB, CE; then it is manifeſt, that each triangle ACB, ACE will anſwer the given conditions; for the angle A and AC are two of the data, and common to each of the triangles, and CB, CE are each equal to PQ the other given ſide. This caſe is therefore ambiguous; except, 1ſt, when the circle touches Am, in which caſe, CE and CB coincide, and become perpendicular to Am. 2dly, when B falls below A, or in mA produced downwards, or when the ſine of A is greater than the ſine of the angle oppoſite to AC, in which caſe, the triangle ACE alone anſwers the given conditions; and the angle AEC is leſs than 90°. When the circle neither cuts nor touches Am, the caſe is impoſſible. The ſame appears from the propoſition; for CB, or CE, : CA :: ſin. A : ſin. of the angle oppoſite to AC; but the ſine of an angle, and the ſine of it's ſupplement, are the ſame (63); therefore there are two angles oppoſite to AC which will anſwer the given conditions, one of which is the ſupplement of the other; let therefore ABC be the greater of theſe angles, and AEC will be the other; for the angle $CEB=CBE$, which

FIG. 7.

which is the ſupplement of *CBA**; and the angle *ABC* is greater, and *AEC* leſs than 90°. When the ſine of the angle oppoſite to *AC* becomes radius, than *CB*, *CE* coincide. When the ſine of the ſame angle becomes leſs than the ſine of the angle *A*, *B* falls below *A*, and the triangle *ACE* alone anſwers the conditions; and when the proportion gives the ſine of the angle oppoſite to *AC* greater than radius, the caſe is impoſſible. *Secondly*, let the angle *A* be *equal* to, or *greater* than a right angle. Here there can be no ambiguity, as the angle oppoſite to the baſe muſt be leſs than a right angle, otherwiſe the three angles of a triangle would be greater than two right angles. In this caſe, *B* muſt always fall below *A*.

Prop. XVI.

If a perpendicular be let fall from the vertex of a triangle upon the baſe, the baſe : the ſum of the other two ſides :: the difference of thoſe ſides : the difference of the ſegments of the baſe.

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* In like manner, whenever we get the ſine of an angle, that angle will be ambiguous, ſo far as we can determine from it's ſine. But in Spherical Trigonometry, the known affections of the ſides and angles of a triangle, will frequently take away the ambiguity, as will afterwards appear. The known limits alſo of the values of quantities in practice, will often prevent a caſe from being doubtful. For inſtance, if by any computation we deduce the ſine of the ſun's true altitude, although that ſine may anſwer to two angles, one as much above as the other is below 90°, yet, as the ſun's altitude can never be greater than 90°, we know that the angle under 90° is that which the caſe requires.

131. Let ABC be a triangle, and BD be perpendicular to AC; with the center B, and radius BA (the leſs of the two ſides), deſcribe a circle, and produce CB to meet it at E; then $CE = CB + BE = CB + BA$ the *ſum* of the ſides; and $CF = CB - BF = CB - BA$ the *difference* of the ſides; alſo, BD biſects AG (Euc. B. III. p. 3.); therefore $CG = CD - DG = CD - DA$ the difference of the ſegments of the baſe. Now $CA \times CG = CE \times CF$ (Euc. B. III. p. 36. Cor.); hence, $CA : CE :: CF : CG$. FIG. 8.

132. Cor. Hence, if the three ſides of a triangle be given, the angles may be found; for CG being found, $\frac{1}{2}(AC + CG) = CD$, and $\frac{1}{2}(AC - CG) = AD$, will be known. Therefore in the right angled triangle BDA, we know BA, AD; hence, we can find the angles BAD, ABD (125); and in the right angled triangle BDC, we know BC, DC; therefore we can find the angles BCD, CBD (125); thus we find the angles of the given triangle. But when the three ſides are given, the angles may be more readily found by the following Propoſition.

Prop. XVII.

As the rectangle under any two ſides of a triangle : $\frac{1}{2}$ (third ſide + difference of the other two ſides) × $\frac{1}{2}$ (third ſide − difference of the other two ſides) :: the ſquare of radius : the ſquare of the ſine of half the angle comprehended between the before-mentioned two ſides.

133. By Euc. B. II. p. 13. $BC^2 = AC^2 + AB^2 - 2AC \times AD = \left(\text{as } AD = \frac{AB \times \text{coſ. } A}{r} \; (125)\right) AC^2 + AB^2 - \frac{2\,AC \times AB \times \text{coſ. } A}{r}$, therefore $r \times \frac{AC^2 + AB^2 - BC^2}{2AC \times AB} = \text{coſ. } A = r - \frac{2 \times \overline{\text{ſin. } \frac{1}{2} A}^2}{r}$ (93); hence, $2 \times \overline{\text{ſin. } \frac{1}{2} A}^2 = r^2 + r^2 \times \frac{BC^2 - AC^2 - AB^2}{2AC \times AB} = r^2 \times \frac{BC^2 - AC^2 - AB^2 + 2AC \times AB}{2AC \times AB} = \frac{r^2 \times (BC + AC - AB) \times (BC + AB - AC)}{2AC \times AB} =$ (if $D = AC - AB$) $\frac{r^2 \times (BC + D) \times (BC - D)}{2AC \times AB}$; therefore $AC \times AB : \frac{1}{2}(BC + D) \times \frac{1}{2}(BC - D) :: r^2 : \overline{\text{ſin. } \frac{1}{2} A}^2$.

The logarithmic calculation of the value of half the angle A is hence very ſimple; for as $\text{ſin. } \frac{1}{2} A = \sqrt{\frac{r^2 \times \frac{1}{2}(BC + D) \times \frac{1}{2}(BC - D)}{AC \times AB}}$, we have (6, 7, 40), $\text{log. ſin. } \frac{1}{2} A = \frac{1}{2}\left(2 \text{ log. } r + \text{log. } \frac{1}{2}(BC + D) + \text{log. } \frac{1}{2}(BC - D) + \text{ar. co. log. } AC + \text{ar. co. log. } AB, \text{ rejecting 20 from the index}\right)$; but $2 \times \text{log. } r = 20{,}0000000$; we may therefore leave out 2 log. r, and that will take 20 from the index; hence, $\text{log. ſin. } \frac{1}{2} A = \frac{1}{2}\left(\text{log. } \frac{1}{2}(BC + D) + \text{log. } \frac{1}{2}(BC - D) + \right.$

ar. co. log. AC+ar. co. log. $AB\Big)$; therefore we get the angle A when the three ſides are given; and there is here no ambiguity, as half A muſt be leſs than 90°.

134. Cor. Hence, coſ. $\frac{1}{2}A = \sqrt{r^2 - \overline{\text{ſin.}\ \tfrac{1}{2}A}^2} =$

$$\sqrt{r^2 - r^2 \times \frac{\frac{1}{2}(BC+AB-AC) \times \frac{1}{2}(BC+AC-AB)}{AC \times AB}}$$

$$= \sqrt{r^2 \times \frac{AC \times AB - \frac{1}{2}(BC+AB-AC) \times \frac{1}{2}(BC+AB-AC)}{AC \times AB}}$$

$$= \sqrt{\frac{r^2 \times \frac{1}{2}(AC+AB+BC) \times \frac{1}{2}(AC+AB-BC)}{AC \times AB}};$$

therefore log. coſ. $\frac{1}{2}A = \frac{1}{2}\Big($log. $\frac{1}{2}(AC+AB+BC)$ +log. $\frac{1}{2}(AC+AB-BC)$+ar. co. log. AC+ar. co. log. $AB\Big)$.

Prop. XVIII.

In any triangle, the ſum of any two ſides : their difference :: the tangent of half the ſum of the two angles oppoſite to thoſe ſides : the tangent of half their difference.

135. Let ABC be any triangle; with the center B FIG.
and radius BC, the leſs of the two ſides AB, BC, de- 9
ſcribe a circle, cutting AB in G, and produce AB to meet the circle in F; join FC, and GC, and draw GN perpendicular to GC. Now as GCF is a right angle (Euc. B. iii. p. 31), GN is parallel to FC (Euc. B. i. p. 27), and the triangles AGN, AFC are ſimilar; alſo, GN is the tangent of ACG, and FC the

the tangent of BGC, to the radius GC (69). Again, $AF=AB+BF=AB+BC$ the *ſum* of the ſides, and $AG=AB-BG=AB-BC$ the *difference* of the ſides. Alſo, $\frac{1}{2}(BCA+BAC)=\frac{1}{2}FBC$ (Euc. B. I. p. 32.) $=BGC$ (Euc. B. III. p. 20.) $=BCG$; and $ACG=BCA-BCG=BCA-\frac{1}{2}(BCA+BAC)=\frac{1}{2}(BCA-BAC)$. And by ſimilar triangles AGN, AFC, $AF : AG :: FC : GN$, that is, $AB+BC : AB-BC ::$ tan. $\frac{1}{2}(BCA+BAC)$: tan. $\frac{1}{2}(BCA-BAC)$.

136. Or, the propoſition may be thus proved. By Prop. 15. $BA : BC ::$ ſin. BCA : ſin. BAC, therefore $BA+BC : BA-BC ::$ ſin. $BCA+$ ſin. BAC : ſin. $BCA-$ ſin. BAC :: (112) tan. $\frac{1}{2}(BCA+BAC)$: tan. $\frac{1}{2}(BCA-BAC)$.

137. Cor. The two ſides AB, BC, and the included angle ABC being given, we have $BCA+BAC=180°-ABC$; therefore (S) half the ſum of the other two angles is known; hence, the three firſt terms in the proportion being known, we get the fourth term (D), half the difference of the ſame two angles; therefore $S+D=$ the greater angle BCA, and $S-D=$ the leſs angle BAC*; thus we find the other two angles. Hence (128), ſin. BAC : ſin. ABC :: BC : AC the third ſide.

138. If an arc be found in terms of it's coſine, and the arc be very ſmall, or very near 180°, the variation of the coſine will be ſo ſmall, that it will not vary in the Tables for many ſeconds. For inſtance, ſuppoſe the log. coſine of a required arc comes out 9,999998; now in the Tables, this is the coſine of an arc from

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* For $\frac{1}{2}(BCA+BAC)+\frac{1}{2}(BCA-BAC)=BCA$, and $\frac{1}{2}(BCA+BAC)-\frac{1}{2}(BCA-BAC)=BAC$.

2′. 52″ to 3′. 41″; here is therefore an arc of 49″ which has the ſame log. coſine in the Tables, owing to their being continued to 7 places of decimals only; it is impoſſible therefore to ſay what arc between 2′. 52″ and 3′. 41″ we are to take. In ſuch caſes, we muſt expreſs it by a ſine. Now if A be any arc, $r - \text{coſ.}\, A = \frac{2 \times \overline{\text{ſin.}\, \frac{1}{2} A}^2}{r}$ (93); therefore $\text{ſin.}\, \frac{1}{2} A = \sqrt{\frac{r^2 - r \times \text{coſ.}\, A}{2}}$; and as the log. ſine of a ſmall arc increaſes faſt, we ſhall thus get A true to a ſecond. In the caſe of Prop. 13, let us ſuppoſe the angle A to be very ſmall. Here, $\text{coſ.}\, A = r \times \frac{AB}{AC}$; therefore $\text{ſin.}\, \frac{1}{2} A = \sqrt{r^2 \times \frac{AC - AB}{2AC}}$. In like manner, if an arc very near 90° be expreſſed by it's ſine, upon account of the ſmall variation of the ſine, the arc can only be known within certain limits; we muſt therefore expreſs it in terms of the coſine. For the methods of proceeding in all ſuch caſes, ſee Dr. MASKELYNE's Introduction to TAYLOR's Logarithms.

PROP. XIX.

Given the three ſides of a triangle, to find it's area.

FIG. 14.

139. By EUC. B. II. p. 13. $BC^2 = AB^2 + AC^2 - 2AD \times AB$, therefore $AD = \frac{AB^2 + AC^2 - BC^2}{2AB}$; and (EUC. B. I. p. 47.) $CD^2 = AC^2 - \frac{\overline{AB^2 + AC^2 - BC^2}|^2}{4AB^2}$

=

$$= \frac{2AB^2 \times AC^2 + 2AB^2 \times CB^2 + 2AC^2 \times CB^2 - AB^4 - AC^4 - BC^4}{4AB^2};$$

hence, the ſquare of the area $(= \frac{1}{4} AB^2 \times CD^2) =$

$$\frac{2AB^2 \times AC^2 + 2AB^2 \times CB^2 + 2AC^2 \times CB^2 - AB^4 - AC^4 - BC^4}{16} =$$

$\frac{1}{2}(AB+AC+BC) \times \frac{1}{2}(AB+AC-BC) \times \frac{1}{2}(AB+BC-AC) \times \frac{1}{2}(AC+BC-AB)$; the area of a triangle is therefore found by the following

Rule. Add the three ſides together, and take half the ſum, and from the half ſum ſubtract each ſide ſeparately; multiply the half ſum and the three remainders together, and the ſquare root of the product is equal to the area.

Prop. XX.

To find the length of a ſtraight object ſtanding perpendicular to the horizon.

Fig. 10. 140. Let PQ be an object, *acceſſible* from A the place of obſervation. Upon a ſtand Am fix a quadrant at m, and let mr be parallel to the horizon, which line may be determined by the teleſcope of the quadrant, if it be an aſtronomical one, by ſetting o on the vernier to o on the limb, if the quadrant ſhow altitudes, or to 90° on the limb, if it ſhow zenith diſtances; and if Hadley's quadrant be uſed, the line may be determined by a level at m. Obſerve the angles Qmr, Pmr, with the quadrant; and meaſure the diſtance mr, by placing ſeveral ſtaffs whoſe tops are in that line, and then meaſure upon them with rods; for the ground will probably not be horizontal, and ſufficiently plain for the purpoſe of meaſuring. Hence, rad. : tan. $Qmr :: mr : Qr$ (123). And to find Pr, we have, rad. : tan. $Pmr :: mr : Pr$; hence, we get $PQ=Qr+Pr$.

If PQ be *inacceſſible*, place the quadrant at ſome other point n in the line mr, and obſerve alſo the angle Qnr, and meaſure the diſtance mn. Then as we know the angle Qnr, we know Qnm it's ſupplement; and having obſerved the angle Qmr, we get $mQn=180°-Qnm-Qmn$; hence (128), ſin. nQm : ſin. $Qmn :: mn : Qn$; and (125) rad. : ſin. $Qnr :: Qn : Qr$. And to find Pr, obſerve the angle Pmr, and we have, tan. Qmr : tan. $Pmr :: Qr : Pr$ (71); hence, we get $Qr+Pr=PQ$.

Or the altitude may be found thus. Let $Qv=$ the radius of the Tables $=$ unity, and draw vsw perpendicular to QP; then (69) vw is the tangent of the angle wQv, or the cotangent of Qmr, and sv is the cotangent of Qnr; therefore sw is the difference of the cotangents of Qmr, Qnr, the two obſerved angles; and by ſimilar triangles, $ws : mn :: Qs : Qn :: Qv\ (=1) : Qr=\frac{mn}{ws}$. In like manner we get Pr. Hence, if $a=$ the difference of the cotangents of Qmr, Qnr, and $b=$ the difference of the cotangents of Pmr, Pnr, we have $\frac{mn}{a}+\frac{mn}{b}=mn\times\frac{a+b}{ab}=PQ$. This is the ſhorteſt operation in practice.

141. Cor. Hence, we can find the length of any part QS from the top, by obſerving, in the firſt caſe, the angle Smr, and then computing Sr as Qr was computed; or, in the ſecond caſe, by obſerving alſo

the angle Snr, and then calculating Sr in the manner in which $\mathcal{Q}r$ was calculated. Hence, we get $\mathcal{Q}r-Sr=\mathcal{Q}S$. Thus we may find the height of a tower ſituated upon a hill.

PROP. XXI.

To carry on a meaſurement by a ſeries of triangles.

FIG. 11. 142. Meaſure a baſe AB, and let C, D be two objects which can be ſeen from A and B. With a theodolite, or any other inſtrument with which you can meaſure horizontal angles, meaſure the angles BAC, ABC, BAD, ABD; then in the two triangles CAB, DAB, we know, in each, the ſide AB, and the angles at A and B, therefore we know ACB the ſupplement of $CAB+CBA$, and ADB the ſupplement of $DAB+DBA$; hence (128), ſin. ACB : ſin. CAB :: AB : BC, and ſin. ADB : ſin. DAB :: AB : BD; thus we obtain BC, BD; and knowing the angles ABC, ABD, we know their difference CBD; therefore in the triangle CBD, we know CB, CD, and the included angle CBD; hence (137), we can find CD, and the angles BCD, BDC. Therefore we know the length of CD; alſo it's diſtance from AB, and it's poſition in reſpect to AB. In like manner, if E, F, be two objects viſible from C and D, we can determine from C and D, the length of EF, it's diſtance from CD, and it's poſition in reſpect to CD. Thus the meaſurement may be continued to any diſtance. And to make the concluſion more accurate, the menſuration from one baſe to another, may be carried on by different ſets of triangles. For inſtance, two objects P, Q, might have been choſen, inſtead of C, D, and the meaſurement of EF might have been obtained by means of them, in like manner. And by taking ſeveral ſeries of triangles which conduct to the ſame two objects, the mean of the reſults will probably be more accurate than any one would have been. After carrying on a ſeries of triangles for ſome diſtance, it is cuſtomary actually to meaſure again the interval of two objects whoſe diſtance has been determined by calculation, in order to find the error of the calculated diſtance. This is called a *Baſe of Verification.*

143. A ſurvey of *England* is at preſent carrying on from a baſe firſt meaſured upon *Hounſlow Heath.* By continuing the meaſurement to *Saliſbury Plains*, the diſtance of two objects was there found, by calculation, to be 36574,4 feet, from the mean reſult of ſeveral ſeries of triangles; and by an actual meaſurement, the diſtance was found to be 36574,3 feet, differing very little more than an inch from the computed diſtance. A proof of the extreme accuracy of the inſtruments, and goodneſs of the obſervations.

144. If there be any number of objects viſible from A and B, the diſtance and poſition of them all may be determined in like manner. Thus a ſurvey may be made of a large town, or city, by taking the poſition of all it's principal buildings at any two ſtations from which they can be ſeen. The poſition and diſtances of ſhips at ſea, and, in general, of any inacceſſible objects, may be determined in like manner.

145. Alſo, by the bye, any extent of land may be meaſured, by meaſuring a baſe, and dividing the

whole into triangles. For the area of the triangle $ACB = \frac{1}{2} AB \times BC \times \text{ſin.}\ CBA$, radius being unity*; the area of $BCD = \frac{1}{2} CB \times BD \times \text{ſin.}\ CBD$; the area $CDF = \frac{1}{2} CD \times DF \times \text{ſin.}\ CDF$; the area $CEF = \frac{1}{2} EC \times GF \times \text{ſin.}\ ECF$; and thus we may proceed for all the triangles into which the whole is divided. This is the moſt ready method, particularly when the ſurveys are large.

FIG. 12. 146. But if one ſide ABC of a field be a curve, let ADC be the chord; and divide ADC into any number of equal parts; in the preſent inſtance, we will ſuppoſe it to be divided into *five*; but the greater the number, the more accurate will be the reſult; and meaſure the perpendiculars mn, pq, rs, tv. Then, conſidering the parts of the curve intercepted between the ſucceſſive perpendiculars to be ſtraight lines, and that Am, mp, pr, rt, tC, are each $\frac{1}{5}$ of AC, we have

$$\begin{aligned}
\text{Area } Amn &= \tfrac{1}{2}\, mn \times \tfrac{1}{5} AC \\
\text{——}\ mnqp &= \tfrac{1}{2}\, (mn + pq) \times \tfrac{1}{5} AC\dagger \\
\text{——}\ pqsr &= \tfrac{1}{2}\, (pq + rs) \times \tfrac{1}{5} AC \\
\text{——}\ rsvt &= \tfrac{1}{2}\, (rs + tv) \times \tfrac{1}{5} AC \\
\text{——}\ Ctv &= \tfrac{1}{2}\, tv \times \tfrac{1}{5} AC \\
\hline
\therefore \text{ area } ABCDA &= (mn + pq + rs + tv) \times \tfrac{1}{5} AC =
\end{aligned}$$

mn+

FIG. 14. * That this is the area may be thus proved. Draw CD perpendicular to AB; then, rad. $= 1 : \text{ſin.}\ B :: BC : DC = BC \times \text{ſin.}\ B$; and the area $ABC = \frac{1}{2} AB \times DC = \frac{1}{2} AB \times BC \times \text{ſin.}\ B$.

FIG. 13. † For let pq, mn be perpendicular to mp, join nq, draw na parallel to mp, and biſect aq in c. Then the area $mnap = mp \times mn$; the area $naq = na \times \frac{1}{2} qa = mp \times ac$; therefore the area $mnqp = mp \times mn + mp \times ac = mp \times (mn + ac) = mp \times pc = mp \times \frac{1}{2} (pq + pa) = \frac{1}{2} (pq + mn) \times mp$.

$\frac{mn+pq+rs+tv}{5} \times AC$. And thus it appears in general, that any curvilinear area *ABCDA* is found neatly, *by adding together all the perpendiculars, dividing the ſum by the number of perpendiculars increaſed by unity, and multiplying by the chord of the curve.*

PROP. XXII.

Given the diſtance CD of two objects, to find the diſtance AB of two obſervers at A and B.

147. Meaſure the angles *CAD*, *CAB*, *DBC*, *DBA*. Now as there are not ſufficient data in any of the triangles to compute the other parts, we muſt aſſume a value for *AB*, and then compute the value of *CD*, as in the laſt Propoſition; then, as the computed value of *CD* is to it's true value, ſo is the aſſumed value of *AB* to it's true value. For the angles at *A* and *B* remaining the ſame, by changing the value of *AB*, the whole figure will continue ſimilar to itſelf, and therefore *AB* will vary as *CD*.

PROP. XXIII.

To meaſure the length of an arc of a meridian.

148. Let *FM* be the meridian of any place *F*; draw *Da*, *Bc* perpendicular to it, and *Db* perpendicular to *Bc*. At *F*, obſerve the ſun's altitude when it is vertical to *D*, and compute it's azimuth *DFA*; then (125), rad. : coſ. *DFa* :: *FD* : *Fa*. Hence, we FIG. 11.

we get the poſition of all the triangles in reſpect to the meridian. For knowing the angles FDB, FDa, aDB $(=90°)$, we know the angle BDb; hence, rad. : coſ. BDb :: BD : $Db=ac$; therefore we get $Fa+ac=Fc$. Thus the meaſures are all reduced to the meridian. Now to determine the difference of latitudes of B and F, find, from obſervation, the true zenith diſtance of a ſtar at each place, and the difference of thoſe zenith diſtances will be the difference of latitudes of B and F. But as the perpendicular Bc does not accurately coincide with a circle of latitude, the latitude of c is not exactly the ſame as that of B. For the utmoſt accuracy therefore, this difference muſt be computed and applied to Fc, and you get the true diſtance of the two places in latitude; hence, the difference of latitudes of B and F : 1° :: the length of the meridian correſponding to the former : the length of a degree. A degree, correſponding to the mean radius of the earth, is 69,2 Engliſh miles. See my *Complete Syſtem of Aſtronomy*, Vol. ii. Art. 1009.

If there be more ſtations between the firſt and the laſt, the difference of latitudes of each may be computed in like manner.

149. Having explained the principles of Plane Trigonometry, we proceed to ſhow the method of reſolving the various caſes of triangles, by logarithms; in which computations, by the ſine, coſine, &c. of an arc, or angle, we mean log. ſine, log. coſine, &c. unleſs the contrary be expreſſed. And in making trigonometrical calculations, it will ſave much time, when the ſame angles occur, to take out all their logarithms at once, as it will avoid the trouble of turning to the

ſame angle again. The computer therefore, before he begins his operation, ſhould put it down in it's proper order, leaving it to be filled up by the logarithms; and he will then ſee what angles are repeated in the operation, and, at one opening of the Tables, he may take out all the logarithms belonging to the ſame angle, and put them down in their proper places.

EXAMPLES.

Ex. 1. *In a right angled triangle ABC, given the baſe $AB = 20$, and the perpendicular $BC = 30$; to find the angles, and the hypothenuſe.* FIG. 6.

By Prop. 12. the baſe (20) : perp. (30) :: rad. : tan. of the angle A. Hence (50), 10,(log. rad.) + log. 30 − log. 20 = log. tan. $\angle A$.

10, + log. 30 - - - - 11,4771213
log. 20 - - - - - 1,3010300

tan. 56°. 18′ = $\angle A$ - - 10,1760913

Hence, 90° − 56°. 18′ = 33°. 42′ = $\angle C$.

Again (128), ſin. A = 56°. 18′ : rad. :: BC = 30 : AC; hence, 10, + log. 30 − log. 56°. 18′ = log. AC.

10, + log. 30 - - - - - 11,4771213
ſin. 56°. 18′ - - - - 9,9200994

log. 36,96 = AC - - - 1,5570219

Or, AC may be immediately found by Art. 126.

Ex. 2. *Given the hypothenuſe $AC=50$, and the baſe $AB=20$; to find the angles.*

By Prop. 13. $AC=50 : AB=20 :: \text{rad.} : \text{coſ.}\, A$; hence, $10, + \text{log.}\ 20 - \text{log.}\ 50 = \text{log. coſ.}\ A$.

10, + log. 20 - - - - - -	11,3010300
log. 50 - - - - - -	1,6989700
coſ. 66°. 25′ = ∠*A* - -	9,6020600

Hence, $90^{\circ} - 66^{\circ}.\ 25' = 23^{\circ}.\ 35' = \angle C$.

Ex. 3. *Given $AB=50$, and the angle $A=25^{\circ}.\ 17'$; to find BC.*

By Prop. 15. ſin. $C = (90^{\circ} - 25^{\circ}.\ 17')\ 64^{\circ}.\ 43' :$ ſin. $A = 25^{\circ}.\ 17' :: AB = 50 : BC$. Hence (49), ar. co. log. ſin. $64^{\circ}.\ 43'$ + log. ſin. $25^{\circ}.\ 17'$ + log. 50 − 10, = log. BC.

Ar. co. ſin. 64°. 43′ - - - - -	0,0437322
ſin. 25. 17 - - - - -	9,6305243
log. 50 - - - - - -	1,6989700
log. 23,617 - - - -	1,3732265

FIG. 7. Ex. 4. *In an oblique angled triangle, given the angle $A=56^{\circ}.\ 14'$, $AC=20$, and the ſide oppoſite to the angle $A=30$; to find the other angles and ſide.*

By Prop. 15. 30 (the ſide oppoſite to the $\angle A$) : $AC = 20$ :: ſin. $56^{\circ}.\ 14'$: ſin. of the angle oppoſite to AC; hence (49), ar. co. log. 30 + log. 20 + log. ſin. $56^{\circ}.\ 14' - 10$, = log. ſin. ∠opp. AC.

Ar.

Ar. co. log. 30	8,5228787
log. 20	1,3010300
ſin. 56°. 14′	9,9197619
ſin. 33. 39 = ∠op. AC	9,7436706

As the angle A is leſs than 90°, and the ſine of the angle A is greater than the ſine of the angle oppoſite to AC, the caſe is not ambiguous, and AEC is the triangle, having the angle AEC leſs than 90° (130).

Hence, the angle $C = 180° - 56°. 14' - 33°. 39' = 90°. 7'$.

Alſo (128), ſin. 56°. 14′ : ſin. 90°. 7′ :: 30 : AE; and ſin. 90°. 7′ is the ſame as ſin. 89°. 53′ (63), which latter we find in the Tables; hence (49), ar. co. log. ſin. 56°. 14′ + log. ſin. 90°. 7′ + log. 30 − 10, = log. AE.

Ar. co. ſin. 56°. 14′	0,0802381
ſin. 90. 7	9,9999991
log. 30	1,4771213
log. 36,088 = AE	1,5573585

Ex. 5. *Given the three ſides, $AB = 40$, $BC = 50$, $AC = 60$; to find the angle A.*

By Prop. 17. the operation will ſtand thus;

Ar. co. log. $AC=60$	8,2218487
Ar. co. log. $AB=40$	8,3979400
$D=20$	1,5440680
$\frac{1}{2}D=10$	1,1760913
$\frac{1}{2}BC=25$	
log. $\frac{1}{2}(BC+D)=35$	2)19,3399480
log. $\frac{1}{2}(BC-D)=15$	
ſin. $\frac{1}{2}A=27°.\ 53'$	9,6699740

Hence, the angle $A=55°.\ 46'$; and here can be no ambiguity, notwithſtanding the quantity comes out in terms of a ſine, becauſe $\frac{1}{2}A$ muſt always be leſs than 90°. In like manner, each of the other angles may be found. Or, one angle being found, the other may be found by Prop. 15. for $BC : AC ::$ ſin. A : ſin. ABC, and $180°-A-ABC=ACB$.

Ex. 6. *In the triangle ABC, given $AB=40$, $BC=70$, and the included angle $ABC=76°.14'$; to find the other parts.*

By Prop. 18. we have $70+40\ (110) : 70-40\ (30) ::$ tan. $51°.\ 53'\ (\frac{1}{2}.\overline{BAC+BCA})$: tan. $\frac{1}{2}.(BAC-BCA)$. Hence (49), ar. co. log. $110+$ log. $30+$ log. tan. $51°.\ 53'-10$, $=$ log. tan. of the fourth term.

Ar. co. log. 110	7,9586073
log. 30	1,4771213
tan. 51°. 33′	10,1053683
tan. 19. 10	9,5410969

Hence, 51°. 53′+19°. 10′=71°. 3′= the angle *BAC*, and 51°. 53′−19°. 10′= 32°. 43′ = the angle *BCA*. And (128), ſin. 32°. 43′ : ſin. 76°. 14′ :: 40. : *AC*. Hence (49), ar. co. log. ſin. 32°. 43′ + log. ſin. 76°. 14′+log. 40−10,=log. *AC*.

Ar. co. ſin. 32°. 43′ - - -	0,2672163
ſin. 76. 14 - - -	9,9873413
log. 40 - - - -	1,6020600
log. *AC*=71,88158 - -	1,8566176

In theſe computations, we have taken out the angles only to the neareſt minute, as they are found in the common Tables of logarithms; but if they be required to ſeconds, they may be found by proportion. For inſtance, in the 5th example, the log. ſine of $\frac{1}{2}A$ is 9,6699740, the neareſt to which in the Tables is 9,6699420, anſwering to 27°. 53′. Now the difference of theſe two logarithms is 320; and in the Tables it is found that the difference of the logarithms correſponding to 1″ is 39,8; hence, 39,8 : 320 :: 1″ : 8″; therefore $\frac{1}{2}A$=27°. 53′. 8″. In like manner, we may find the angle correſponding to any logarithm, to ſeconds, by the common Tables. By TAYLOR's Logarithms, the angle can be taken out to ſeconds at once. The following examples we ſhall compute to ſeconds.

EX. 7. *Given mn* = 40 *yards, the angle Qmn* = 31°. 17′. 18″, *the angle Qnr* = 48°. 19′. 47″, *and the angle Pmr*=7°. 8′. 17″; *to find PQ.* FIG. 10.

The angle $\mathcal{Q}nm = 180^\circ - 48^\circ.\ 19'.\ 47'' = 131^\circ.\ 40'.\ 13''$, and the angle $m\mathcal{Q}n = 180^\circ - 31^\circ.\ 17'.\ 18'' - 131^\circ.\ 40'.\ 13'' = 17^\circ.\ 2'.\ 29''$; hence, by Prop. 15. and Art. 49, 52.

Ar. co. ſin. 17°. 2'. 29''	0,5330398
ſin. 31. 17. 18	9,7154561
log. 40	1,6020600
log. 70,885 = $\mathcal{Q}n$	1,8505559

ſin. 48°. 19'. 47''	9,8733108
log. 70,885	1,8505559
log. 52,95 = $\mathcal{Q}r$	1,7238667

Ar. co. tan. 31°. 17'. 18''	0,2162888
tan. 7. 8. 17	9,0977123
log. 52,95	1,7238667
log. 10,911 = Pr	1,0378678

Hence, $P\mathcal{Q} = 52,95 + 10,911 = 63,861$.

FIG. 11. Ex. 8. *Given* $AB = 117$ *yards, the angle* $CAB = 70^\circ.\ 18'$, *the angle* $CBA = 50^\circ.\ 21'$, *the angle* $DAB = 47^\circ.\ 51'$, *and the angle* $DBA = 104^\circ.\ 17'$; *to find CD.*

The angle $ACB = 180^\circ - 70^\circ.\,18' - 50^\circ.\,21' = 59^\circ.\,21'$, and $ADB = 180^\circ - 47^\circ.\ 51' - 104^\circ.\ 17' = 27^\circ.\ 52'$; alſo, $DBC = 104^\circ.\ 17' - 50^\circ.\ 21' = 53^\circ.\ 56'$; hence, by Prop. 15. and Art. 49.

Ar.

Ar. co. ſin. 59°. 21′ - - - - - -	0.0653514
ſin. 70. 18 - - - - - - -	9,9738067
log. 117 - - - - - - - -	2,0681859
log. 128,04 = *BC* - - -	2,1073440

Ar. co. ſin. 27°. 52′ - - - - -	0,3302968
ſin. 47. 51 - - - - - - -	9,8700470
log. 117 - - - - - - - -	2,0681859
log. 185,58 = *BD* - - - - -	2,2685297

Now the angles $BCD + BDC = 180° - 53°. 56' = 126°. 4'$; hence, by Art. 135, $185,58 + 128,04 = 313,62 : 185,58 - 128,04 = 57,54 ::$ tan. 63°. 2′ (½. 126°. 4′) : tan. ½ $(BCD - BDC)$;

Ar. co. log. 313,62 - - - - - - -	7,5035963
log. 57,54 - - - - - - -	1,7599699
tan. 63°. 2′. 0″ - - - - -	10,2934590
tan. 19. 49. 45 = ½$(BCD - BDC)$	9,5570252

Hence, 63°. 2′ + 19°. 49′. 45″ = 82°. 51′. 45″ = the angle *BCD*, and 63°. 2′ − 19°. 49′. 45″ = 43°. 12′. 15″ = the angle *BDC*. And by Prop. 15. and Art. 49.

Ar. co. ſin. 82°. 51′. 45″ - - - 0,0033785
ſin. 53. 56. 0 - - - 9,9075901
log. 185,58 - - - - 2,2685297

log. 151,18 - - - - 2,1794983

Thus we may find the length and poſition of any line which is inacceſſible.

Ex. 9. *Given the angles CAB, CBA, DAB, DBA, as in the laſt Example, and the diſtance CD = 200 yards; to find AB.*

By Prop. 22. aſſume *AB* of ſome value, ſuppoſe 117 yards; then, by the laſt Example, *CD* = 151,18 yards; but it's true value is 200; hence, 151,18 : 200 :: 117 : the true value of *AB*; by Art. 49. therefore,

Ar. co. log. 151,18 - - - - - 7,8205016
log. 200 - - - - - - 2,3010300
log. 117 - - - - - - 2,0681859

log. 154,78 = *AB* - - - 2,1897175

Ex. 10. *Given the three ſides of a triangle* 40, 50, *and* 60; *to ſind it's area.*

By Articles 139, 4, 7, the operation by logarithms is as ſollows:

	40	
	50	
	60	
Sum	150	
½ Sum	75	log. 1,8750613
3 Rem^{s}.	35	log. 1,5440680
	25	log. 1,3979400
	15	log. 1,1760913
		2)5,9931606
Area=992,1569		log. 2,9965803

◆

APPENDIX.

APPENDIX.

PROP. XXIV.

TO expreſs the powers of the ſine of an arc, in terms of the ſine or coſine of the multiples of the arc, aſſuming radius $=1$.

150. From Art. 107. if we make $B=A$, we get $\overline{\text{ſin. } A}^2 = \frac{1}{2} - \frac{1}{2}\text{ coſ. } 2A$.

Multiply each ſide of the laſt equation by ſin. A, and we have, $\overline{\text{ſin. } A}^3 = \frac{1}{2}\text{ ſin. } A - \frac{1}{2}\text{ coſ. } 2A \times \text{ſin. } A$; but (105) $\frac{1}{2}\text{ coſ. } 2A \times \text{ſin. } A = \frac{1}{4}\text{ ſin. } 3A - \frac{1}{4}\text{ ſin. } A$; hence, $\overline{\text{ſin. } A}^3 = \frac{1}{2}\text{ ſin. } A - \frac{1}{4}\text{ ſin. } 3A + \frac{1}{4}\text{ ſin. } A = \frac{3}{4}\text{ ſin. } A - \frac{1}{4}\text{ ſin. } 3A$.

Multiply the laſt equation by ſin. A, and we have, ſin. $A^4 = \frac{3}{4}\overline{\text{ſin. } A}^2 - \frac{1}{4}\text{ ſin. } 3A \times \text{ſin. } A$; but (by the firſt ſtep of this Propoſition) $\frac{3}{4}\overline{\text{ſin. } A}^2 = \frac{3}{8} - \frac{3}{8}\text{ coſ. } 2A$; and (107) $\frac{1}{4}\text{ ſin. } 3A \times \text{ſin. } A = \frac{1}{8}\text{ coſ. } 2A - \frac{1}{8}\text{ coſ. } 4A$; hence, $\overline{\text{ſin. } A}^4 = \frac{3}{8} - \frac{3}{8}\text{ coſ. } 2A - \frac{1}{8}\text{ coſ. } 2A + \frac{1}{8}\text{ coſ. } 4A = \frac{3}{8} - \frac{4}{8}\text{ coſ. } 2A + \frac{1}{8}\text{ coſ. } 4A$.

In like manner we may proceed as far as we pleaſe, by multiplying the laſt equation by ſin. A, and then reſolving the products on the right hand ſide by Art. 105. and 107.; hence, we get,

$$\text{ſin.}\ A = \text{ſin.}\ A.$$
$$\overline{\text{ſin.}\ A}^2 = \tfrac{1}{2} - \tfrac{1}{2}\ \text{coſ.}\ 2A.$$
$$\overline{\text{ſin.}\ A}^3 = \tfrac{3}{4}\ \text{ſin.}\ A - \tfrac{1}{4}\ \text{ſin.}\ 3A.$$
$$\overline{\text{ſin.}\ A}^4 = \tfrac{3}{8} - \tfrac{4}{8}\ \text{coſ.}\ 2A + \tfrac{1}{8}\ \text{coſ.}\ 4A.$$
$$\overline{\text{ſin.}\ A}^5 = \tfrac{10}{16}\ \text{ſin.}\ A - \tfrac{5}{16}\ \text{ſin.}\ 3A + \tfrac{1}{16}\ \text{ſin.}\ 5A.$$
$$\overline{\text{ſin.}\ A}^6 = \tfrac{10}{32} - \tfrac{15}{32}\ \text{coſ.}\ 2A + \tfrac{6}{32}\ \text{coſ.}\ 4A - \tfrac{1}{32}\ \text{coſ.}\ 6A.$$
$$\overline{\text{ſin.}\ A}^7 = \tfrac{35}{64}\ \text{ſin.}\ A - \tfrac{21}{64}\ \text{ſin.}\ 3A + \tfrac{7}{64}\ \text{ſin.}\ 5A - \tfrac{1}{64}\ \text{ſin.}\ 7A.$$

&c. &c. &c.

The law of continuation is manifeſt; the odd powers are expreſſed in terms of the ſines, and the even powers in terms of the coſines, of the multiples of A; the ſigns of the terms are alternately + and −; the numerators of the coefficients, reckoning from right to left, are the coefficients of a binomial whoſe power is the ſame as that of ſin. A, except in the even powers, where that term in which A does not enter, has the numerator of it's coefficient only one half of the correſponding coefficient of the binomial; and the denominators are that power of 2 whoſe index is leſs by unity than that of ſin. A.

Prop. XXV.

To expreſs the powers of the coſine of an arc, in terms of the coſine of the multiples of the arc, ſuppoſing radius $= 1$.

151. From Art. 106. if we make $B=A$, we get, $\overline{\text{coſ.}\ A}^2=\frac{1}{2}+\frac{1}{2}\ \text{coſ.}\ 2A$.

Multiply each ſide of the laſt equation by coſ. A, and we have, $\overline{\text{coſ.}\ A}^3=\frac{1}{2}\ \text{coſ.}\ A+\frac{1}{2}\ \text{coſ.}\ 2A\times\text{coſ.}\ A$; but (106) $\frac{1}{2}\ \text{coſ.}\ 2A\times\text{coſ.}\ A=\frac{1}{4}\ \text{coſ.}\ 3A+\frac{1}{4}\ \text{coſ.}\ A$; hence, $\overline{\text{coſ.}\ A}^3=\frac{1}{2}\ \text{coſ.}\ A+\frac{1}{4}\ \text{coſ.}\ 3A+\frac{1}{4}\ \text{coſ.}\ A=\frac{3}{4}\ \text{coſ.}\ A+\frac{1}{4}\ \text{coſ.}\ 3A$.

Multiply each ſide of the laſt equation by coſ. A, and we have, $\overline{\text{coſ.}\ A}^4=\frac{3}{4}\ \overline{\text{coſ.}\ A}^2+\frac{1}{4}\ \text{coſ.}\ 3A\times\text{coſ.}\ A$; but (by the firſt ſtep of this Propoſition) $\frac{3}{4}\ \overline{\text{coſ.}\ A}^2=\frac{3}{8}+\frac{3}{8}\ \text{coſ.}\ 2A$; and (106) $\frac{1}{4}\ \text{coſ.}\ 3A\times\text{coſ.}\ A=\frac{1}{8}\ \text{coſ.}\ 4A+\frac{1}{8}\ \text{coſ.}\ 2A$; hence, $\overline{\text{coſ.}\ A}^4=\frac{3}{8}+\frac{3}{8}\ \text{coſ.}\ 2A+\frac{1}{8}\ \text{coſ.}\ 4A+\frac{1}{8}\ \text{coſ.}\ 2A=\frac{3}{8}+\frac{4}{8}\ \text{coſ.}\ 2A+\frac{1}{8}\ \text{coſ.}\ 4A$. Thus we may proceed as far as we pleaſe, by multiplying the laſt equation by coſ. A, and reſolving the products on the right hand ſide by Art. 106; hence, we get,

$$\text{coſ.}\ A=\text{coſ.}\ A.$$
$$\overline{\text{coſ.}\ A}^2=\tfrac{1}{2}+\tfrac{1}{2}\ \text{coſ.}\ 2A.$$
$$\overline{\text{coſ.}\ A}^3=\tfrac{3}{4}\ \text{coſ.}\ A+\tfrac{1}{4}\ \text{coſ.}\ 3A.$$
$$\overline{\text{coſ.}\ A}^4=\tfrac{3}{8}+\tfrac{4}{8}\ \text{coſ.}\ 2A+\tfrac{1}{8}\ \text{coſ.}\ 4A.$$
$$\overline{\text{coſ.}\ A}^5=\tfrac{10}{16}\ \text{coſ.}\ A+\tfrac{5}{16}\ \text{coſ.}\ 3A+\tfrac{1}{16}\ \text{coſ.}\ 5A.$$
$$\overline{\text{coſ.}\ A}^6=\tfrac{10}{32}+\tfrac{15}{32}\ \text{coſ.}\ 2A+\tfrac{6}{32}\ \text{coſ.}\ 4A+\tfrac{1}{32}\ \text{coſ.}\ 6A.$$
$$\overline{\text{coſ.}\ A}^7=\tfrac{35}{64}\ \text{coſ.}\ A+\tfrac{21}{64}\ \text{coſ.}\ 3A+\tfrac{7}{64}\ \text{coſ.}\ 5A+\tfrac{1}{64}\ \text{coſ.}\ 7A.$$

&c. &c. &c.

The law of continuation is the ſame as in the laſt Propoſition, only all the terms are here poſitive.

PROP. XXVI

To conſtruct a Table of ſines, coſines, verſed-ſines, tangents, cotangents, ſecants, and coſecants, for every minute of the quadrant.

152. If z be the arc of a circle, whoſe radius $=1$, it's ſine $=z-\frac{z^3}{2\cdot3}+\frac{z^5}{2\cdot3\cdot4\cdot5}-$ &c. (*Prin. Flux.* Prop. 103.; hence, the arc $-$ the ſine $=\frac{z^3}{2\cdot3}-\frac{z^5}{2\cdot3\cdot4\cdot5}+$ &c. Now if the ſemiperiphery 3,14159265358979 &c. be divided by 10800 (the number of minutes in 180°) the quotient is 0,000290888 &c. the length of an arc of 1 minute. If therefore we ſubſtitute this value for z, it will be found that the difference between the arc of 1′ and it's ſine is 0,0000000000041 very nearly; therefore we may aſſume 0,000290888 the ſine of 1′.

Or, the ſine of 1 minute may be obtained thus. By Prop. 4. if a be an angle, ſin. $\frac{1}{2}a=\sqrt{\frac{1}{2}\text{ ver. ſin. }a}$. Now if $a=30°$, ſin. $a=\frac{1}{2}$ (91), and (76), coſ. 30° $=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$; hence (77), ver. ſin. $30°=1-\frac{\sqrt{3}}{2}=\frac{2-\sqrt{3}}{2}$; therefore ſin. $15°=\sqrt{\frac{2-\sqrt{3}}{4}}$. Find the coſine, and thence the verſed-ſine of 15°; hence, we get the ſine of 7°. 30′. Thus we may continue to biſect the arc, and find it's ſine; and after 12 biſections we come to an arc of $52''.\ 44'''.\ 3''''.\ 45'''''$, whoſe ſine is known; but (as appears from above) ſmall arcs are very nearly equal to their ſines, and conſequently will be to each other as their ſines, very nearly; hence, $52''.\ 44'''.\ 3'''' : 45''''' : 1' ::$ the ſine of the former : the ſine of the latter. Having obtained the ſine of 1′, find it's coſine (76), and then the ſines of all the arcs above 1′ may be found in the following manner.

FIG. 4. Let $AP=a$, $PM=b$, then $AM=a+b$, $AN=a+2b$; and by ſimilar triangles, CMH, NPE, $CM : CH :: PN : NE$, that is, $1 : \text{coſ.}\,(a+b) :: 2\,\text{ſin.}\,b : \text{ſin.}\,(a+2b)-\text{ſin.}\,a$; hence, $\text{ſin.}\,(a+2b)=2\,\text{coſ.}\,(a+b)\times\text{ſin.}\,b+\text{ſin.}\,a$. Let therefore $b=1'$, $a=0'$, $1'$, $2'$, $3'$, $4'$, &c. and we have,

$$\text{ſin.}\,2'=2\,\text{coſ}\,1'\times\text{ſin.}\,1'.$$
$$\text{ſin.}\,3'=2\,\text{coſ.}\,2'\times\text{ſin.}\,1'+\text{ſin.}\,1'.$$
$$\text{ſin.}\,4'=2\,\text{coſ.}\,3'\times\text{ſin.}\,1'+\text{ſin.}\,2'.$$

And in general,

$$\text{ſin.}\,n'=2\,\text{coſ.}\,\overline{n-1}'\times\text{ſin.}\,1'+\text{ſin.}\,\overline{n-2}'.$$

Hence, the ſine and coſine of 1′ being known, we get the ſine of 2′; therefore we can find it's coſine (76); hence, we get the ſine of 3′; proceed thus, and find the ſine and coſine for every minute of the arc up to 30°.

Now by Art. 104. $\text{ſin.}\,(30°+B)=\text{coſ.}\,B-\text{ſin.}\,(30°-B)$; take therefore $B=1'$, $2'$, $3'$, $4'$, &c. and we have,

$$\text{ſin.}\,30°.\,1'=\text{coſ.}\,1'-\text{ſin.}\,29°.\,59'$$
$$\text{ſin.}\,30.\,2=\text{coſ.}\,2-\text{ſin.}\,29.\,58$$
$$\text{ſin.}\,30.\,3=\text{coſ.}\,3-\text{ſin.}\,29.\,57$$

&c. &c.

Find, in this manner, all the ſines, and thence all the coſines (76), as far as 45°; hence, we have found all the ſines and coſines as far as 90°; for (66) the ſine and coſine of an arc above 45° are reſpectively equal to the coſine and ſine of an arc as much below 45°, one of theſe arcs being the complement of the other; that is, the coſine of 44° = ſine of 46°; the coſine of 43° = ſine of 47°; &c. Alſo, the ſine of 44° = coſine of 46°; the ſine of 43° = coſine of 47°; &c.

Having determined the ſines and coſines for all the arcs to 90°, we get the verſed-ſines (77). Find (78) the tangents up to 45°; and then the tangents from 45° to 90° may be found, by addition, by the following theorem: Tan. $(45° + B)$ = tan. $(45° - B)$ + 2 tan. $2B$*; by writing therefore for B, 1′, 2′, 3′, 4′, &c. we get the tangents of 45°. 1′, 45°. 2′, 45°. 3′, 45°. 4′, &c. up to 90°. And having got the tangents, the cotangents are known, the tangent of an arc being the cotangent of the complement of that arc (72); thus, the tangent of 50° = the cotangent of 40°.

By Prop. 8. if A = an arc, coſec. $2A = \frac{1}{2}$ (tan. A + cotan. A); if therefore we take A = 1′, 2′, 3′, 4′, &c.

---

* By Prop, 10. if $A = 45°$, $T = 1$; and we have, tan. $(45° + B) = \frac{1+t}{1-t}$, and tan. $(45° - B) = \frac{1-t}{1+t}$; therefore, tan. $(45° + B)$ — tan. $(45° - B) = \frac{1+t}{1-t} - \frac{1-t}{1+t} = \frac{4t}{1-t^2} = \left(\text{as } 1 - t^2 = \frac{2t}{\text{tan. } 2B} \ (118)\right)$ 2 tan. $2B$; hence, tan. $(45° + B)$ = tan. $(45° - B)$ + 2 tan. $2B$.

If $B = 45°$, we have, tan. 90° = 2 tan. 90°, which is another inſtance of a falſe concluſion ariſing from the ſuppoſition that an arc of 90° has a tangent.

we get the coſecants of 2′, 4′, 6′, 8′, &c. by taking half the ſum of two known quantities; and the coſecants of 1′, 3′, 5′, 7′, &c. may be very readily found by interpolation. And the coſecants up to 90° being found, the ſecants of the ſame arcs will be known, the ſecant of an arc being the coſecant of it's complement (73); thus, the coſecant of 35° is the ſecant of 55°.

Having obtained the natural ſines, coſines, &c. if their logarithms be found from the common Tables of logarithms of the natural numbers, and 10 be added to the index, you have the logarithms of the ſines, coſines, &c. as found in the Tables; for the logarithms are there ſuppoſed to be logarithms of the ſines, coſines, &c. to radius 10000000000.

In computing the Tables of quantities to any number of figures to be put down, the computation ſhould extend to two figures more, in order to put down the laſt figure to it's neareſt value. Thus, if it be required to put down ſeven figures, you ſhould compute to nine figures, and then the two laſt figures will always ſhow the neareſt to be put down in the ſeventh place; thus, if the computed value be 847962356, it's neareſt value to ſeven figures is 8479624. If the figure in the eighth place be not a 5, eight figures will be ſufficient; for if the eighth figure be above or below five, the figure in the ſeventh place muſt be a unit above or below the value found in the computation.

PROP.

Prop. XXVII.

Given an arc z, to find it's ſine and coſine in terms of the impoſſible quantity $\sqrt{-1}$.

153. Put x = coſ. of the arc z, and let radius be unity; then $\dot{z} = \frac{-\dot{x}}{\sqrt{1-x^2}}$, therefore $\dot{z}\sqrt{-1} = \frac{-\dot{x}\sqrt{-1}}{\sqrt{1-x^2}} = \frac{\dot{x}}{\sqrt{x^2-1}}$, whoſe fluent is $z\sqrt{-1}$ = hyp. log. $\overline{x+\sqrt{x^2-1}}$; put e = the number whoſe h. l. is unity; then (6) h. l. $e^{z\sqrt{-1}} = z\sqrt{-1}$ = h. l. $\overline{x+\sqrt{x^2-1}}$; ∴ $e^{z\sqrt{-1}} = x+\sqrt{x^2-1}$; tranſpoſe x and ſquare both ſides, and we get $e^{2z\sqrt{-1}} - 2xe^{z\sqrt{-1}} = -1$; hence, $\frac{e^{z\sqrt{-1}}+e^{-z\sqrt{-1}}}{2} = x$ = *coſine* of z. Alſo, $x^2 = \frac{e^{2z\sqrt{-1}}+2+e^{-2z\sqrt{-1}}}{4}$, therefore $x^2-1 = \frac{e^{2z\sqrt{-1}}-2+e^{-2z\sqrt{-1}}}{4}$, and $1-x^2 = \frac{e^{2z\sqrt{-1}}-2+e^{-2z\sqrt{-1}}}{-4}$; hence, $\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{2\sqrt{-1}} = \sqrt{1-x^2}$ = *ſine* of z.

154. COR. 1. If for z we write nz, we get,

$$\text{Coſ. } nz = \frac{e^{nz\sqrt{-1}} + e^{-nz\sqrt{-1}}}{2}.$$

$$\text{Sin. } nz = \frac{e^{nz\sqrt{-1}} - e^{-nz\sqrt{-1}}}{2\sqrt{-1}}.$$

Theſe expreſſions for the ſine and coſine of any arc z, or of it's multiple, in terms of imaginary quantities, are of great uſe in phyſical Aſtronomy; ſee my *Complete Syſtem of Aſtronomy*, Vol. ii. Art. 1147.

155. COR. 2. Becauſe $\text{coſ. } z = \dfrac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2}$

$$\text{And } \sqrt{-1} \times \text{ſin. } z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2}$$

$$\therefore\ \text{coſ. } z + \sqrt{-1} \times \text{ſin. } z = e^{z\sqrt{-1}}$$

$$\text{And coſ. } z - \sqrt{-1} \times \text{ſin. } z = e^{-z\sqrt{-1}}.$$

Hence (154),

$$\text{coſ.}\, nz = \frac{\overline{\text{coſ.}\, z + \sqrt{-1} \times \text{ſin.}\, z}\,\rceil^{n} + \overline{\text{coſ.}\, z - \sqrt{-1} \times \text{ſin.}\, z}\,\rceil^{n}}{2}$$

$$\text{ſin.}\, nz = \frac{\overline{\text{coſ.}\, z + \sqrt{-1} \times \text{ſin.}\, z}\,\rceil^{n} - \overline{\text{coſ.}\, z - \sqrt{-1} \times \text{ſin.}\, z}\,\rceil^{n}}{2\sqrt{-1}}.$$

PROP.

PROP. XXVIII.

To expreſs an arc in terms of the ſines of multiples of that arc.

156. The hyp. log. of $1+x=x-\frac{1}{2}x^2+\frac{1}{3}x^3-$ &c.

And, hyp. log. of $1+\frac{1}{x}=x^{-1}-\frac{1}{2}x^{-2}+\frac{1}{3}x^{-3}-$ &c.

Now hyp. log. $\overline{1+x}$ − hyp. log. $\overline{1+\frac{1}{x}\left(\frac{1+x}{x}\right)}$ = hyp. log. $\overline{1+x}$ − hyp. log. $\overline{1+x}$ + hyp. log. x (5) = hyp. log. x; hence,

Hyp. log. $x=x-x^{-1}-\frac{1}{2}(x^2-x^{-2})+\frac{1}{3}(x^3-x^{-3})-$ &c.

Put $e^{z\sqrt{-1}}$ for x; then (6) hyp. log. $x=z\sqrt{-1}\times$ hyp. log. $e=z\sqrt{-1}$; hence, $\frac{1}{2}z=\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{2\sqrt{-1}}$ $-\frac{1}{2}\left(\frac{e^{2z\sqrt{-1}}-e^{-2z\sqrt{-1}}}{2\sqrt{-1}}\right)+\frac{1}{3}\left(\frac{e^{3z\sqrt{-1}}-e^{-3z\sqrt{-1}}}{2\sqrt{-1}}\right)$ − &c. But (154) $\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{2\sqrt{-1}}$ = ſin. z, $\frac{e^{2z\sqrt{-1}}-e^{-2z\sqrt{-1}}}{2\sqrt{-1}}$ = ſin. $2z$, $\frac{e^{3z\sqrt{-1}}-e^{-3z\sqrt{-1}}}{2\sqrt{-1}}$ = ſin. $3z$, &c. Hence, $\frac{1}{2}z$ = ſin. $z-\frac{1}{2}$ ſin. $2z+\frac{1}{3}$ ſin. $3z$ − &c.

157. If we make z = an arc of 90°, then ſin. $2z$, ſin. $4z$, &c. become = 0, and ſin. $z=1$, ſin. $3z=-1$, ſin. $5z=1$, &c. hence, an arc of $45°=1-\frac{1}{3}+\frac{1}{5}-$ &c.

154. Cor. 1. *If for z we write nz,* we get,

$$\text{Coſ. } nz = \frac{e^{nz\sqrt{-1}} + e^{-nz\sqrt{-1}}}{2}.$$

$$\text{Sin. } nz = \frac{e^{nz\sqrt{-1}} - e^{-nz\sqrt{-1}}}{2\sqrt{-1}}.$$

Theſe expreſſions for the ſine and coſine of any arc z, or of it's multiple, in terms of imaginary quantities, are of great uſe in phyſical Aſtronomy; ſee my *Complete Syſtem of Aſtronomy*, Vol. ii. Art. 1147.

155. Cor. 2. Becauſe $\text{coſ. } z = \frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2}$

$$\text{And } \sqrt{-1} \times \text{ſin. } z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2}$$

$$\therefore \text{ coſ. } z + \sqrt{-1} \times \text{ſin. } z = e^{z\sqrt{-1}}$$

$$\text{And coſ. } z - \sqrt{-1} \times \text{ſin. } z = e^{-z\sqrt{-1}}.$$

Hence (154),

$$\text{coſ. } nz = \frac{\overline{\text{coſ.} z + \sqrt{-1} \times \text{ſin.} z}\,\vert^{n} + \overline{\text{coſ.} z - \sqrt{-1} \times \text{ſin.} z}\,\vert^{n}}{2}$$

$$\text{ſin. } nz = \frac{\overline{\text{coſ.} z + \sqrt{-1} \times \text{ſin.} z}\,\vert^{n} - \overline{\text{coſ.} z - \sqrt{-1} \times \text{ſin.} z}\,\vert^{n}}{2\sqrt{-1}}$$

PROP. XXVIII.

To expreſs an arc in terms of the ſines of multiples of that arc.

156. The hyp. log. of $1+x = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 -$ &c.
And, hyp. log. of $1+\frac{1}{x} = x^{-1} - \frac{1}{2}x^{-2} + \frac{1}{3}x^{-3} -$ &c.

Now hyp. log. $\overline{1+x}$ − hyp. log. $\overline{1+\frac{1}{x}}\left(\frac{1+x}{x}\right)$
= hyp. log. $\overline{1+x}$ − hyp. log. $\overline{1+x}$ + hyp. log. x (5)
= hyp. log. x; hence,
Hyp. log. $x = x - x^{-1} - \frac{1}{2}(x^2 - x^{-2}) + \frac{1}{3}(x^3 - x^{-3}) -$ &c.
Put $e^{z\sqrt{-1}}$ for x; then (6) hyp. log. $x = z\sqrt{-1}$ × hyp.
log. $e = z\sqrt{-1}$; hence, $\frac{1}{2}z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}}$
$-\frac{1}{2}\left(\frac{e^{2z\sqrt{-1}} - e^{-2z\sqrt{-1}}}{2\sqrt{-1}}\right) + \frac{1}{3}\left(\frac{e^{3z\sqrt{-1}} - e^{-3z\sqrt{-1}}}{2\sqrt{-1}}\right)$
− &c. But (154) $\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}}$ = ſin. z,
$\frac{e^{2z\sqrt{-1}} - e^{-2z\sqrt{-1}}}{2\sqrt{-1}}$ = ſin. $2z$, $\frac{e^{3z\sqrt{-1}} - e^{-3z\sqrt{-1}}}{2\sqrt{-1}}$ =
ſin. $3z$, &c. Hence, $\frac{1}{2}z$ = ſin. $z - \frac{1}{2}$ ſin. $2z + \frac{1}{3}$ ſin.
$3z -$ &c.

157. If we make z [illegible] f $90°$, [illegible] ſin. $2z$,
ſin. $4z$ = &c. become = [illegible] . $z = 1$ [illegible] $= -1$,
ſin. $5z$ = hence [illegible] f $45°$ [illegible] − &c.

SPHERICAL

TRIGONOMETRY.

DEFINITIONS, &c.

Art. 158. A *GREAT* circle of a ſphere is that whoſe plane paſſes through the center of the ſphere. A *ſmall* circle is that whoſe plane does not paſs through the center*.

159. Cor. Hence, all great circles of a ſphere biſect each other. For each paſſing through the center of the ſphere, the common ſection of any two, being a ſtraight line (Euc. B. xi. p. 3.), muſt be a diameter; and a diameter of a circle biſects it. Hence, the interſections of the peripheries of two great circles are at two points diametrically oppoſite to each other.

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* It will be proved in Prop. 1. that every ſection of a ſphere is a circle.

160. A ſpherical triangle is formed by the arcs of three great circles upon the ſurface of a ſphere.

FIG. 15. 161. If *ABDE* be a great circle of a ſphere whoſe center is *C*, and *PCp* be a diameter of the ſphere perpendicular to that circle, the points *P*, *p*, are called the *Poles* of that great circle. And if the ſmall circle *abcd* be perpendicular to *PC*, we ſhall call *P*, *p*, the *Poles* of that ſmall circle.

162. COR. Hence, if the pole of a great circle be given, that circle is given; for the circle muſt paſs through the center of the ſphere, and be perpendicular to a radius drawn to it's pole. And if the pole and one point of a ſmall circle be given, that circle will be given; for the circle muſt paſs through that point, and be perpendicular to a radius of the ſphere, drawn to it's pole.

163. Great circles *PAp*, *PFp*, paſſing through the poles *P*, *p*, of the great circle *ABDE*, are called *Secondaries* to that circle.

164. When the arcs of circles are mentioned, they are underſtood to be arcs of great circles, the radii of which are the radii of the ſphere; for in Trigonometry, the ſines, coſines, &c. are always ſuppoſed to refer to arcs of equal radii.

PROP. I.

Any ſection abcd of a ſphere is a circle.

165. For let *PrC* be perpendicular to *abcd*, cutting it in *r*, and join *rd*, *Cd*; then *rd* being perpendicular to *rC*, we have $rd = \sqrt{Cd^2 - Cr^2}$; now *Cr* is conſtant for the ſame ſection, and *Cd* is alſo conſtant, being

the radius of the ſphere; hence, rd is conſtant; therefore the ſection is a circle, whoſe center is r.

166. Cor. Hence, $Pd = \sqrt{Pr^2 + rd^2}$ is conſtant; therefore all the arcs of great circles intercepted between P and the periphery $abcd$ are equal (Euc. B. III. p. 28.).

Prop. II.

The poles of a great circle are 90° diſtant upon the ſurface of the ſphere from every point of the periphery of that circle.

167. For draw any great circle PFp, cutting the circumference $ABDE$ in F, and join FC; then PCF being a right angle, the arc $PF = 90°$ (60). For the ſame reaſon, $pF = 90°$.

168. Cor. Hence, if about two points upon the ſurface of a ſphere which are 180° diſtant, two great circles be drawn, they will coincide.

Prop. III.

The periphery of a ſecondary is at right angles to the periphery of it's great circle, at the point of their interſection.

169. The direction of the periphery of a circle at any point being the ſame as the direction of a tangent at that point, the angle CFP is a right angle (Euc. B. III. p. 16. Cor.); alſo, PCF is a right angle; and the arc PF is in the plane FCP; therefore the direction of the periphery PF at F muſt be parallel to PC (Euc. B. I. p. 28.); but PC is perpendicular to

the circle *ABDF*; therefore *PF* at *F* is perpendicular to the circle *ABDE* (Euc. B. XI. p. 8.); hence, the arc *PF* at *F* is at right angles to *AF* at *F* (Euc. B. XI. Def. 3. and p. 13.). For the ſame reaſon, *PAF* is a right angle *.

170. Cor. 1. Hence, and by Art. 167, if a great circle *PFp* be perpendicular to *ABDE*, and *FP*, *Fp*, be taken each $=90°$, *P*, *p* are the poles of *ABDE*. And if any two great circles *PAp*, *PFp*, be perpendicular to *ABFD*, they meet at the poles *P*, *p*, of that circle.

171. Cor. 2. As *PAp*, *PFp*, are perpendicular to *ABDF*, if $AE=90°$, *E* is the pole of *PAp*; and if $FH=90°$, *H* is the pole of *PFp*; and ſince $AE=FH$; take *FE* from each, and we have $EH=AF$; that is, the poles of two great circles lie in a great circle which is perpendicular to each of thoſe great circles; and the diſtance between the poles is equal to the diſtance between the two great circles deſcribed about them, meaſured upon a great circle perpendicular to them †.

PROP.

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* In ſpherical Trigonometry, we frequently ſpeak of one circle being perpendicular to another, in reference only to the peripheries of thoſe circles, in which caſe, we mean that the peripheries are perpendicular to each other at their interſection.

† If *PAp* be the equator, *PFp* the ecliptic, *E* and *H* their reſpective poles; then the great circle *EH* paſſing through the poles, cuts the ecliptic and the equator at right angles, and paſſes through the tropic *F*; and *EH* the diſtance of the poles is equal to *AF* the ſun's greateſt declination.

PROP. IV.

If two points be aſſumed any where in the periphery of a great circle, and another point be aſſumed at the diſtance of 90° *from each of thoſe points, that point muſt be one of the poles of that great circle.*

172. Let G, F be any two points in the periphery of the great circle $ABDE$, the poles of which are P, p; and if poſſible let $Gx = Fx = 90°$. Join xC; then as $Gx = 90°$, GCx is a right angle; and as $Fx = 90°$, FCx is a right angle; therefore xC is perpendicular to GC and FC; hence, it is perpendicular to the plane GCF (Euc. B. XI. p. 4.); but pC is perpendicular to the plane GCF; therefore x muſt coincide with p, otherwiſe two lines would be perpendicular to the ſame plane at the ſame point, which is impoſſible (Euc. B. XI. p. 13). It muſt however be obſerved, that there is one exception to this, which is, when the two aſſumed points are diametrically oppoſite, or when $GF = 180°$, in which caſe, the two circles deſcribed about F and G as poles will coincide (168), and therefore every point in that circle is 90° from F and G (167).

PROP. V.

The angle formed by the peripheries of two great circles on the ſurface of a ſphere, is equal to the angle formed by the planes of thoſe circles; and is meaſured by the arc of a great circle intercepted between them, deſcribed about their interſection as a pole.

173. At P the peripheries FP, AP, are perpendicular to the common interſection PCp of the planes of the circles, therefore the angle FPA is the angle between the planes (Euc. B. xi. Def. 6.); and as FC, AC are alſo perpendicular to PCp in the planes PFp, PAp, they are reſpectively parallel to the directions of the peripheries FP, AP, at P (Euc. B. i. p. 28.); therefore the angle FCA is equal to the angle at P formed by the peripheries FP, AP (Euc. B. xi. p. 10.); and the angle FCA is meaſured by the arc AF, which is the arc of a great circle whoſe pole is P. For the ſame reaſon, the angle ApF is equal to ACF. Hence, $APF = ApF$.

174. Cor. 1. Hence, when two planes interſect each other, their inclination is every where the ſame; for wherever C may be aſſumed, FCA is, in like manner, proved equal to FPA.

175. Cor. 2. By Art. 171. the arc AF meaſures the diſtance of the poles of PAp, PFp; therefore the diſtance of the poles of the two great circles meaſures the inclination of the planes of thoſe circles.

176. Cor. 3. If the ſmall circle $abcd$ be parallel to $ABDE$, the arcs ad, AF, comprehended between the great circles $PaAp$, $PdFp$, contain the ſame number of degrees. For the angle $ACF = ard$ (174). Hence, the arc ad is in proportion to the angle APF; for the angle $APF = ACF$, which is meaſured by AF, which contains the ſame number of degrees as ad.

177. Cor. 4. Let $PeGp$ be another great circle; then $ad : ae :: \angle APF : \angle APG$ (176); conceive now the circle $abceda$ to move from P to p, then $ad : ae$ always in the ſame ratio; therefore, by compoſition,

ſition, the ſum of all the *ad*'s : ſum of all the *ae*'s in the ſame ratio; that is, the area $PApFP$: area $PApGP$ :: $\angle APF$: $\angle APG$.

178. Cor. 5. As the ſectors ard, ACF are ſimilar, $AF : ad :: AC : ar$: rad. :: ſin. aP (62), or coſ. Aa (66). Hence, as AF meaſures the angle APF, we have $APF = \frac{\text{rad.} \times ad}{\text{ſin. } aP} = \text{coſec. } aP \times ad$ (83), if radius be unity. Alſo, $ad = \frac{\text{ſin. } aP \times APF}{\text{rad.}}$.

Prop. VI.

The ſum of any two ſides of a ſpherical triangle is greater than the third ſide.

FIG. 24.

179. Let ABC be a ſpherical triangle, O the center of the ſphere, and join AO, BO, CO; then the three angles AOB, BOC, COA form a ſolid angle at O, of which three angles, the ſum of any two is greater than the third (Euc. B. xi. p. 20.); but the arcs AB, BC, CA, are the meaſures of theſe three angles reſpectively; hence, the ſum of any two of theſe arcs is greater than the third.

180. Cor. As $AB+BC$ is greater than AC, by taking BC from each, AB is found to be greater than $AC-BC$; that is, any one ſide is greater than the difference of the other two ſides.

PROP. VII.

Only one great circle can be drawn through two given points on the ſurface of a ſphere.

181. For a great circle muſt paſs through the center of the ſphere, and three points determine the poſition of a plane.

182. COR. Hence, a great circle cannot be drawn through more than two points any how aſſumed. But a ſmall circle may be drawn through any three points on the ſurface of a ſphere, becauſe a plane may be drawn through any three points.

PROP. VIII.

If two ſpherical triangles have two ſides of one equal to two ſides of the other, and the included angles equal, the two triangles will be equal.

183. The proof of this is the ſame as that for plane triangles in EUC. B. 1. p. 4, except that the third ſides are here proved to coincide by the laſt Propoſition.

PROP. IX.

In an iſoſceles ſpherical triangle, the angles oppoſite to the equal ſides are equal.

184. The proof of this is the ſame as that for plane triangles in EUC. B. 1. p. 5. uſing here the

preceding Propoſition, where EUCLID uſes his fourth Propoſition.

185. COR. Hence, an equilateral triangle is equiangular.

PROP. X.

If a ſpherical triangle have two angles equal, the ſides oppoſite to theſe two angles will be equal.

186. The proof of this is the ſame as that for plane triangles in EUC. B. 1. p. 6.

PROP. XI.

Any ſide of a ſpherical triangle is leſs than 180 *degrees.*

187. Let ABC be a triangle, and produce AB, AC to meet at D, then ABD, ACD, are each 180° (159); therefore AB, AC, are each leſs than 180°. If CB, CA be produced to meet, it appears in like manner that CB is leſs than 180°. FIG. 16.

PROP. XII.

The ſum of the three ſides of a ſpherical triangle is leſs than 360 *degrees.*

188. For $AB+BD+AC+CD=ABD+ACD=360°$; but $BD+CD$ is greater than BC; therefore $AB+AC+BC$ is leſs than 360°.

PROP.

PROP. XIII.

In any ſpherical triangle, the greater angle is oppoſite to the greater ſide.

FIG. 17. 189. Let ABC be a triangle, having the angle ABC greater than the angle A, and make the angle $ABD=A$; then $AD=DB$ (186); add DC to each, and $AC=DB+DC$, which is greater than BC (179).

PROP. XIV.

FIG. 16. *In any ſpherical triangle ABC, if $AB+BC$ be equal to, greater, or leſs than 180 degrees, the internal angle A will be reſpectively equal to, greater, or leſs than the external angle BCD; and $A+BCA$ will be reſpectively equal to, greater, or leſs than two right angles.*

190. *Firſt*, if $AB+BC=180°=ABD$, then $BC=BD$; hence (184), $BCD=BDC=A$ (173).

Secondly, if $AB+BC$ be greater than ABD, then BC is greater than BD; hence, BDC, or A, is greater than BCD (189).

Thirdly, if $AB+BC$ be leſs than ABD, then BC is leſs than BD; hence, BDC, or A, is leſs than BCD (189).

Again, the angle $BCD+BCA=2$ right angles; hence,

Firſt, if $AB+BC=180°=ABD$, then $BD=BC$, and (184) $BCD=BDC=A$ (173); therefore $A+BCA=2$ right angles.

Secondly, if $AB+BC$ be greater than ABD, then BC is greater than BD, and (189) BDC, or A, is greater than BCD; therefore $A+BCA$ is greater than two right angles.

Thirdly, if $AB+BC$ be leſs than ABD, then BC is leſs than BD, and (189) BDC, or A, is leſs than BCD; therefore $A+BCA$ is leſs than two right angles.

PROP. XV.

FIG. 19.

If in a ſpherical triangle ABC, about the three angles A, B, C, as poles, the great circles QR, PR, PQ, be deſcribed, the ſides of PQR will be the ſupplements of the angles ABC, to which they are oppoſite; and the angles of PQR will be the ſupplements of the ſides of ABC to which they are oppoſite.

191. For produce the ſides of ABC, to m, n, s, r, v, w. Now as A is the pole of QR, $AQ=90°$, and as C is the pole of PQ, $CQ=90°$ (167); therefore, as AC is leſs than 180° (187), and AQ and CQ are each 90°, Q is the pole of AC (172). For the like reaſon, P is the pole of BC, and R is the pole of PQ. Hence, $Qm=90°$, and $Pr=90°$; therefore $180°=Qm+Pr=Qm+Pm+mr=QP+mr$; but mr is the meaſure of the angle ACB (173); therefore PQ is the ſupplement of ACB. For the like reaſon, PR is the ſupplement of ABC, and QR is the ſupplement of BAC.

Again, $Cr=90°$, and $Bs=90°$; therefore $180°=Cr+Bs=Cr+Cs+BC=rs+BC$; but rs meaſures the angle P (173); therefore the angle P is the ſupplement of BC. For the like reaſon, the angle $\mathcal{Q}$ is the ſupplement of AC, and the angle R is the ſupplement of AB.

PROP. XVI.

The ſum of the three angles of a ſpherical triangle is greater than two, and leſs than ſix right angles.

192. For (191) $\mathcal{Q}P+mr+\mathcal{Q}R+vn+PR+ws=3\times180°=6\times90°$; therefore $mr+vn+ws$, or (173) $\angle C+\angle A+\angle B$, is leſs than $6\times90°$; but $\mathcal{Q}R+\mathcal{Q}P+PR$ is leſs than $4\times90°$ (188); therefore $mr+vn+ws$, or $\angle C+\angle A+\angle B$, is greater than $2\times90°$.

PROP. XVII.

The ſum of any two angles of a ſpherical triangle is greater than the third angle, by a quantity which is always leſs than 180°.

193. For (191) $180°-P=BC$, $180°-\mathcal{Q}=AC$, $180-R=AB$; therefore (179) $180°-R$ is leſs than $360°-P-\mathcal{Q}$; add $P+\mathcal{Q}-180°$ to each, and we find that $P+\mathcal{Q}-R$ is leſs than 180°.

PROP. XVIII.

In a right angled ſpherical triangle, the ſum of the two angles at the hypothenuſe is greater than 90°, *and leſs than* 270°. *Alſo, the difference of the two angles at the hypothenuſe is leſs than* 90°.

194. For if BAC be a right angle, $QR = 90°$; therefore (179) $PQ+PR$ is greater than 90°, and (180) leſs than 270°; but PR, PQ are the meaſures of the angles B, C; therefore $B+C$ is greater than 90°, and leſs than 270°. Again, $A+B-C$ is leſs than 180° (193); but $A=90°$; therefore $B-C$ is leſs than 90°.

PROP. XIX.

The exterior angle of a ſpherical triangle is leſs than the ſum of the interior and oppoſite angles.

195. For the angle $BCA+BCD=2$ right angles; but $BCA+CBA+CAB$ is greater than 2 right angles (192); therefore BCD is leſs than $CBA+CAB$. FIG. 16.

PROP. XX.

If P be the pole of a great circle BCDEB, and A be any point of the great circle EPB between E and P, and from A, arcs of great circles be drawn to BCDEB; then, of theſe arcs, APB is the greateſt; AE is the leaſt; and of the arcs AC, AD, that is the greater which lies nearer to APB. FIG. 18.

196. Join EB, and it paſſes through the center O (158), and draw AQ perpendicular to EB, and join QC, QD, AC, AD, AE, AB. Now QC is greater than QD (Euc. B. III. p. 7.), and QA being common to the two right angled triangles AQC, AQD, the hypothenuſe AC is greater than AD (Euc. B. I. p. 47); therefore the arc AC is greater than AD. In like manner, as QB is the greateſt and QE the leaſt line that can be drawn to $BCDE$ (Euc. B. III. p. 7.),

AB is the greateſt chord, and AE the leaſt; therefore APB is the greateſt arc, and AE the leaſt.

197. Cor. 1. If one leg of a right angled ſpherical triangle be *greater* than 90°, it is *greater* than the hypothenuſe; and if it be *leſs* than 90°, it is *leſs* than the hypothenuſe. For ABC is a ſpherical triangle, right angled at B (169); and AB is greater than 90°, and alſo greater than AC (196). Again, AEC is a ſpherical triangle, right angled at E; and AE is leſs than 90°, and it is alſo leſs than AC (196).

198. Cor. 2. If one leg of a right angled ſpherical triangle be greater than 90°, as you increaſe the other leg you diminiſh the hypothenuſe; for in the triangle ABC, as BC increaſes, AC decreaſes. But if one leg be leſs than 90°, as you increaſe the other leg you increaſe the hypothenuſe; for in the triangle AEC, as EC increaſes, AC increaſes.

Prop. XXI.

In a right angled ſpherical triangle, the legs are of the ſame affections as the oppoſite angles; that is, if the legs be greater or leſs than quadrants, the angles oppoſite to them will be greater or leſs than right angles.

Fig. 20. 199. Let the angle BAC be a right angle, and AC be a quadrant; then C is the pole of AB (170), and the angle CBA is a right angle (169). Hence, in the right angled triangle BAF, AF is greater than a quadrant, and the oppoſite angle ABF is greater than a right angle; and in the triangle ABG, AG is leſs than a quadrant, and the oppoſite angle ABG is leſs than a right angle.

200. Cor. If either of the legs become a quadrant, the oppoſite angle is a right angle; for the triangle ABC has the leg AC a quadrant, and the oppoſite angle ABC is a right angle.

Prop. XXII.

If the legs, or the angles at the hypothenuſe, of a right angled ſpherical triangle, be of the ſame *affections, the hypothenuſe is* leſs *than a quadrant; and if the legs be of* different *affections, the hypothenuſe is* greater *than a quadrant.*

201. Let A be a right angle, and produce AB, AF to meet at D; then D is a right angle (173); take $AP =$ a quadrant, and join PG; then P is the pole of ACD (170), and PC is a quadrant (167); alſo, C being the pole of AB (170), BC is a quadrant. Now in the right angled triangle ABG, the legs AB, AG are each *leſs* than a quadrant, and the hypothenuſe BG is *leſs* than a quadrant, being leſs than BC (196); and in the right angled triangle DBG, DB, DG are each *greater* than a quadrant, and BG is *leſs* than a quadrant. Again, in the right angled triangle ABF, the leg AB is *leſs*, and AF is *greater* than a quadrant, and the hypothenuſe BF is *greater* than a quadrant, being greater than BC. Now the legs are of the ſame affections as the oppoſite angles (199); therefore if thoſe angles be of the *ſame* or *different* affections, the hypothenuſe will be *leſs* or *greater* than a quadrant.

202. Cor. 1. The converſe of this propoſition is manifeſtly true, that is, if the hypothenuſe be *leſs* or

greater than a quadrant, the legs will be of the *ſame* or *different* affections; for if the hypothenuſe BG be leſs than a quadrant, B and G muſt lie on the ſame ſide of P and C; and if the hypothenuſe BF be greater than a quadrant, B and F muſt lie on different ſides of P and C; becauſe the arc BG is leſs, and BF is greater than BC, which is a quadrant.

203. Cor. 2. If one leg AC be a quadrant, the hypothenuſe BC muſt be a quadrant; and converſely, if the hypothenuſe $BC=90^\circ$, one leg (AC) at leaſt muſt be $=90^\circ$. When two legs AP, AC become quadrants, the third ſide PC is a quadrant, and each angle is a right angle (185).

Prop. XXIII.

FIG. 22. *The ſegments BD, DC of the baſe, and the ſegments BAD, CAD of the vertical angle, are of the* ſame, *or* different *affections, according as the ſides BA, CA are of the* ſame, *or* different *affections.*

204. For (201) if BD, AD, be of the *ſame* affection BA is *leſs* than a quadrant; and if DC, AD, be of the *ſame* affection, AC is *leſs* than a quadrant; alſo, if BD, DA, be of *different* affections, BA is *greater* than a quadrant; and if DC, DA, be of *different* affections, AC is *greater* than a quadrant. Therefore if BD, DC be of the *ſame* affection, BA, AC are of the *ſame* affection. But if BD be of the ſame affection as AD, and DC of a different affection from AD, then AB is leſs than a quadrant, and AC is greater than a quadrant; that is, when BD, DC are

of different affections, *AB*, *AC* are of different affections. But (199) *BD* is of the ſame affection as *BAD*, and *DC* is of the ſame affection as *CAD*; therefore *BAD*, *CAD* are of the *ſame*, or *different* affections, according as *BA*, *CA* are of the *ſame*, or *different* affections.

PROP. XXIV.

In any ſpherical triangle ABC, if the angles ABC, ACB at the baſe be of the ſame *affection, the perpendicular AD from the vertical angle upon the baſe, falls* within *the baſe; if they be of* different *affections, the perpendicular falls* without *the baſe.* FIG. 21, 22.

205. In the firſt caſe, if *AD* do not fall within the baſe, let it fall without; then the angles *ABD*, *ACD* are of the ſame affection, each being of the ſame affection as *AD* (199); but *ACB*, *ACD* are of different affections, one being the ſupplement of the other; hence, *ABD*, *ACB* are of different affections; but, by ſuppoſition, they are of the ſame affection; therefore *AD* does not fall without the baſe. In the ſecond caſe, if *AD* do not fall without the baſe, let it fall within; then the angles *ABD*, *ACD* are of the ſame affection, being of the ſame affection as *AD* (199); but, by ſuppoſition, they are of different affections; therefore *AD* does not fall within the baſe.

The affections of the ſides and angles given in the preceding propoſitions, will frequently take away the ambiguity which would otherwiſe ariſe in the ſolution of ſpherical triangles.

206. Let

FIG. 23. 206. Let *ABC* be a right angled ſpherical triangle, right angled at *B*; produce *BC* to *a*, making $Ba = 90°$, then *a* is the pole of *AB* (170); and with *A* as a pole, deſcribe the arc *abD*, meeting *AB*, *AC* (produced if neceſſary) in *D* and *b*; then *abC* is called the *Complemental* triangle to *ABC*. Now *Ca* is the complement of *CB*; *Cb* is the complement of *AC*; the angle *Cab* is meaſured by *BD* (173), which is the complement of *AB*; and *ba* is the complement of *bD*, which meaſures the angle *A* (173). Alſo, the angle $aCb = ACB$, and the angle *abC* is a right angle (169). And as the ſine or tangent of an arc, or angle, is the coſine or cotangent of their complement (66, 72), the ſine or tangent of *Ca*, *Cb*, *Cab*, and *ba*, is the coſine or cotangent of *CB*, *AC*, *AB*, and the angle *A* reſpectively; and the coſine or cotangent of the former, are reſpectively the ſine or tangent of the latter.

THE SOLUTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

FIG. 24. 207. Let *ABC* be a right angled ſpherical triangle, right angled at *B*, *O* the center of the ſphere, and join *AO*, *BO*, *CO*; then if we conſider the plane *OAB* as coinciding with the plane of the paper, we muſt conceive the plane of *OBC* to be perpendicular to the ſame plane. Draw *CF* perpendicular to *OB*, and *Fm* perpendicular to the ſame line, in the plane *OAB*; then the angle *CFm* meaſures the inclination of the planes *OBC*, *OBA* (Euc. B. xi. Def. 6), therefore *CFm* is a right angle; hence, *CF* being perpendicular to *OFB* and *Fm*, it is perpendicular to the the plane *OBA* (Euc. B. xi. p. 4.). Draw *FE*, *BD* perpendicular to *OA*, and join *CE*, and it will be perpendicular to *AO*. For as *CFE*, *CFO* are right angles, $CE^2 - EF^2 = CF^2 = CO^2 - OF^2$ (Euc. B. i. p. 47.); hence, $CO^2 - CE^2 = OF^2 - EF^2 = OE^2$, therefore *CEO* is a right angle. Hence, as *EC*, *EF* are perpendicular to *AO*, the angle *CEF* is the inclination of the planes *AOC*, *AOB*, and therefore the angle $CEF = CAB$ (173). Alſo, from the conſtruction, *CE* is ſine, and *OE* the coſine of *AC*; *CF* is the ſine, and *OF* the coſine of *CB*; and *BD* is the ſine, and *OD* the coſine of *AB*; alſo, the ſine, coſine and tangent of the angle *CEF* are the ſame as thoſe of *CAB*.

PROP. XXV.

In a right angled ſpherical triangle, rad. : ſine of the hypothenuſe :: ſine of either angle at the hypothenuſe : ſine of it's oppoſite leg.

208. For in the right angled triangle *CFE*, rad. : ſin. *CEF* :: *CE* : *CF* (125); that is, alternando, rad. : ſin. *AC* :: ſin. *CAB* : ſin. *BC*.

In the very ſame manner it may be proved for the angle *C* and it's oppoſite ſide; for if *BC* had been made the baſe, *C* would have been the angle at the baſe; conſequently the angle *ACB*, and *AB*, would have entered the proceſs, inſtead of *CAB*, and *BC*. For the ſame reaſon, in any proportion involving the angle *A*, or *C*, and the ſide *AB*, or *CB*, the terms having ſimilar relations may be ſubſtituted the one for the other.

209. COR. Let A be an *acute* angle, then BC is leſs than 90° (199); and as radius is greater than ſin. AC, except $AC = 90$, ſin. A is greater than ſin. BC, unleſs $AC = 90°$; therefore A is always equal to or greater than BC. Let A be an *obtuſe* angle, then BC is greater than 90° (199); and of two arcs greater than 90°, that which has the greater ſine is the leſs angle (89); therefore A is leſs than BC, unleſs $AC = 90°$. Except therefore when the hypothenuſe is 90°, an acute angle is greater, and an obtuſe angle is leſs, than it's oppoſite ſide.

PROP. XXVI

In a right angled ſpherical triangle, rad. : ſin. of one leg :: tang. of the angle adjacent to that leg : tang. of the other leg.

210.

$$\begin{array}{r l}
\text{For rad.} : \text{tan.}\,CEF, \text{ or } CAB, :: & EF : FC\ (123) \\
\text{tan.}\,BC : \text{rad.} :: & FC : OF\ (123) \\
\hline
\therefore\ \text{tan.}\,BC : \text{tan.}\,CAB :: & EF : OF \\
:: & BD : BO\ (\text{ſim. tri.}) \\
:: & \text{ſin.}\,AB : \text{rad.} \\
\text{or, rad.} : \text{ſin.}\,AB :: & \text{tan.}\,CAB : \text{tan.}\,BC.
\end{array}$$

In like manner it may be proved for the other angle and its adjacent and oppoſite legs.

211. *General properties of right angled ſpherical triangles.*

I. If one leg be *greater* than 90°, it is *greater* than the hypothenuſe; if it be *leſs* than 90°, it is *leſs* than the hypothenuſe (197).

II. The legs are of the ſame affections as the oppoſite angles (199).

III. The two angles at the hypothenuſe are together *greater* than 90°, and *leſs* than 270° (194).

IV. The difference of the two angles at the hypothenuſe is leſs than 90° (194).

V. If the legs, or the angles at the hypothenuſe, be of the *ſame* affection, the hypothenuſe is *leſs* than 90°; if they be of *different* affections, the hypothenuſe is *greater* than 90° (201).

VI. If the hypothenuſe =90°, one at leaſt of the legs =90° (203).

VII. If one of the angles at the hypothenuſe be *acute*, it is *greater* than it's oppoſite ſide; if it be *obtuſe*, it is *leſs* than it's oppoſite ſide; except the hypothenuſe be =90°, in which caſe they are equal (208).

Theſe properties are given by Dr. MASKELYNE, without demonſtration, in his excellent Introduction to TAYLOR's Logarithms.

212. Lord NAPIER, the inventor of logarithms, diſcovered two theorems, by which all the caſes of right angled ſpherical triangles may be ſolved. In a right angled triangle, beſides the right angle, there are five other parts, namely, two angles, the two legs,

FIG. 23. and the hypothenuſe. Now the legs themſelves *AB*, and *BC*, the complements of the two angles *A* and *C*, and the complement of the hypothenuſe *AC*, he calls *Circular Parts*, as they follow one another round the triangle, thus; *BC*, comp. *C*, comp. *CA*, comp. *A*, *AB*; and if any one of theſe be made the *Middle Part*, the two lying next it are called *Adjacent Extremes*; and the other two are called *Oppoſite Extremes*. The following Table ſhows all the caſes.

Caſes	*Middle Part*	*Adjacent Extremes*	*Oppoſite Extremes*
1	*AB*	comp. *A*, *BC*	comp. *C*, com. *AC*
2	*BC*	comp. *C*, *AB*	comp. *A*, com. *AC*
3	comp. *A*	comp. *AC*, *AB*	comp. *C*, *BC*
4	comp. *C*	comp. *AC*, *BC*	comp. *A*, *AB*
5	comp. *AC*	comp. *C*, comp. *A*	*AB*, *BC*

This being premiſed, the two theorems are as follows:

THEO. I. *The rectangle under radius and ſine of the middle part = the rectangle under the tangents of the adjacent extremes.*

THEO. II. *The rectangle under radius and ſine of the middle part = the rectangle under the coſines of the oppoſite extremes.*

The following Table contains all the equations ariſing from theſe theorems, put down in the order in which the parts ſtand in the foregoing Table.

1	Middle Part.	AB	$r \times$ ſin. AB =	cot. $A \times$ tan. BC ſin. $C \times$ ſin. AC.
2		BC	$r \times$ ſin. BC =	cot. $C \times$ tan. AB ſin. $A \times$ ſin. AC
3		comp. A	$r \times$ coſ. A =	cot. $AC \times$ tan. AB ſin. $C \times$ coſ. BC
4		comp. C	$r \times$ coſ. C =	cot. $AC \times$ tan. BC ſin. $A \times$ coſ. AB
5		comp. AC	$r \times$ coſ. AC =	cot. $C \times$ cot. A coſ. $AB \times$ coſ. BC

DEMONSTRATION.

CASES 1, 2. *Make AB the middle part.*

In the triangle ABC, we have,

ſin. AB : tan. BC :: rad. : tan. A (210) :: cot. A : rad. (82);

Hence, rad. × ſin. AB = cot. $A \times$ tan. BC.

Again, by Prop. 25, we have,

rad. : ſin. C :: ſin. AC : ſin. AB;

Hence, rad. × ſin. AB = ſin. $C \times$ ſin. AC.

In like manner the theorems may be proved, if BC be made the middle part.

CASES 3, 4. *Make the complement of the angle A the middle part.*

In the triangle abC, we have,

$$\text{tan. } bC : \text{ſin. } ab :: \text{tan. } a : \text{rad. } (210)\text{; or,}$$

$$\text{cot.}AC : \text{coſ.}A :: \text{cot.}AB : \text{rad.}(206) :: \text{rad.} : \text{tan.}AB\,(82);$$

Hence, rad. × coſ. A = cot. AC × tan. AB.

Again, by Prop. 25, we have,

$$\text{rad.} : \text{ſin. } aC :: \text{ſin. } C : \text{ſin. } ab$$

$$\text{or, rad.} : \text{coſ. } BC :: \text{ſin. } C : \text{coſ. } A\ (206);$$

Hence, rad. × coſ. A = ſin. C × coſ. BC.

In like manner the theorems may be proved, if the complement of the angle C be made the middle part.

CASE 5. *Make the complement of AC the middle part.*

In the triangle aCb, we have,

$$\text{ſin.}bC : \text{tan.}ab :: \text{rad.} : \text{tan.}C\,(210) :: \text{cot.}C : \text{rad.}\,(82);$$

$$\text{or, coſ. } AC : \text{cot. } A :: \text{cot. } C : \text{rad. } (206);$$

Hence, rad. × coſ. AC = cot. C × cot. A.

Again, by Prop. 25, we have,

$$\text{rad.} : \text{ſin. } Ca :: \text{ſin. } a : \text{ſin. } bC$$

$$\text{or, rad.} : \text{coſ. } BC :: \text{coſ. } AB : \text{coſ. } AC\ (206);$$

Hence, rad. × coſ AC = coſ. AB × coſ. BC.

213. By theſe equations, all the caſes of right angled ſpherical triangles may be ſolved; for when any two parts are given, to find a third, that is to be aſſumed the middle part, which will make the other two, either adjacent extremes, or oppoſite extremes. For inſtance, if AB and BC be given, to find A; then if AB be made the middle part, BC and comp. A are the adjacent extremes, and rad. × ſin. $AB =$ cot. A × tan. BC, as in the firſt caſe. Or, without the conſideration of the rule, the equation may be immediately taken from the Table, by obſerving which equation it is that contains the two given and the required quantities. For inſtance, if AB and BC be given, to find AC; then, by Caſe 5, rad. × coſ $AC =$ coſ. AB × coſ. BC; therefore (50) log. coſ. $AC =$ log. coſ. AB + log. coſ. $BC - 10$,. And, in like manner, in any of theſe equations, the log. of one factor = the ſum of the logarithms of the two factors on the contrary ſide, – the log. of the factor on the ſame ſide, or + it's arithmetic complement – 10, (29). The value of the unknown quantity is thus immediately obtained from the equation, without reducing it to a proportion. The reſolution of the equation into an analogy may be uſeful, when you want to ſee the relative values of the quantities.

214. The ambiguous caſes (ariſing from the value of the required quantity being expreſſed by it's ſine) are, when a leg and it's oppoſite angle are given, to find either of the other parts. All other caſes, which from the ſame circumſtance would be ambiguous, have the ambiguity taken away, by ſome of the properties in Art. 211.

215. If we take the ſupplemental triangle to ABC (191), one ſide of it will be a quadrant, and this is called a *quadrantal* triangle; and as the ſine, coſine and tangent of an arc, are the ſame in magnitude as the ſine, coſine and tangent of it's ſupplement, the equations for the triangle ABC will all apply to it's ſupplemental triangle. Hence, a quadrantal triangle may be reſolved by the above rules, conſidering the two angles adjacent to the ſide of 90°, the complement of the other angle, and the complements of the other two ſides, as the circular parts. If therefore FIG. 27. the triangle ZPS have the ſide $ZS = 90°$, we have the circular parts in their order as follows; the angle Z, comp. ZP, comp. P, comp. SP, the angle S. Hence, we have the following equations.

1	Middle Part.	Z	r × ſin. Z =	tan. S × cot. ZP ſin. P × ſin. SP
2		S	r × ſin. S =	tan. Z × cot. SP ſin. P × ſin. ZP
3		comp. ZP	r × coſ. ZP =	tan. Z × cot. P coſ. S × ſin. SP
4		comp. SP	r × coſ. SP =	tan. S × cot. P coſ. Z × ſin. ZP
5		comp. P	r × coſ. P =	cot. SP × cot. ZP coſ. S × coſ. Z

When any two parts are given to find a third, the caſe may be immediately found by inſpection in the Table, obſerving which equation it is that contains the three parts. For inſtance, if SP and ZP be given, to find the angle P, we have, in Caſe 5, $r \times$ coſ. $P =$ cot. $SP \times$ cot. PZ; therefore log. coſ. $P =$ log. cot. $SP +$ log. cot. $PZ - 10$,. The ambiguous caſes are the ſame here as in a right angled triangle, that is, when one ſide and it's oppoſite angle are given. In all other caſes, the ambiguity is taken away by ſome of the following properties of quadrantal triangles.

General Properties of a Quadrantal Triangle.

216. Call the angle oppoſite to the quadrantal ſide, the hypothenuſal angle, the arc of 90° a quadrant, and the other parts, ſimply ſides and angles.

I. If one angle be *greater* than 90°, it is *greater* than the hypothenuſal angle; if *leſs* than 90°, it is *leſs* than the hypothenuſal angle.

II. The angles are of the ſame affections as the oppoſite ſides.

III. The two ſides are together *greater* than 90°, and *leſs* than 270°.

IV. The difference of the two ſides is leſs than 90°.

V. If the angles, or the ſides, be of the *ſame* affection, the hypothenuſal angle is *greater* than 90°; if of *different* affections, it is *leſs* than 90°.

VI. If the quadrantal angle =90°, one at leaſt of the other angles =90°.

VII. If one ſide be *acute*, it is *greater* than it's oppoſite angle; if it be *obtuſe*, it is *leſs* than it's oppoſite angle; except the hypothenuſal angle =90°, in which caſe they are equal.

FIG. 19.

Theſe properties are alſo given by Dr. MASKELYNE, and their proofs follow immediately from the properties of a right angled triangle ABC, by taking it's ſupplemental triangle PQR, in which one ſide PR (oppoſite to the right angle B) $=90°$; for whatever parts are of the ſame or different affections in ABC, muſt have their correſponding ſupplemental parts in PQR of the ſame or different affections; and by ſubſtituting 180° − one angle in PQR for the ſupplemental ſide in ABC, we get the limits of the angles of PQR.

In caſes, where the affection of a required ſide or angle is not known, if the triangle be delineated upon the ſurface of a globe, you may generally diſcover whether the ſide or angle be greater or leſs than 90°.

FIG. 23.

217. Becauſe rad. × coſ. AC = coſ. AB × coſ. BC (212), if AC be leſs than 90°, it's coſine is poſitive, and therefore rad. × coſ. AC, or coſ. AB × coſ. BC, is poſitive; hence, AB, BC, muſt be both leſs, or both greater than 90°; for in the former caſe, their coſines will be poſitive, and in the latter, they will be negative, and therefore in each caſe their product will be poſitive, as it ought. If AC be greater than 90°, it's coſine is negative; therefore coſ. AB × coſ. BC is negative; hence, one of the legs AB, BC, muſt be

leſs than 90°, and the other greater; for then the coſine of one will be negative, and that of the other poſitive, and therefore their product will be negative, as it ought. Theſe concluſions agree with what is proved in Art. 202.

218. Alſo, rad. × ſin. BC = cot. C × tan. AB (212); now ſin. BC is always poſitive, therefore cot. C × tan. AB is always poſitive, conſequently AB and C muſt be both leſs, or both greater than 90°; for in the *former* caſe, the ſigns of cot. C, and tan. AB, will be poſitive, and in the *latter* caſe, they will be negative, and therefore in each caſe their product will be poſitive, as it ought. Hence, the angles are of the ſame affections as the oppoſite ſides. This agrees with what is proved in Art. 199.

The concluſions in theſe two Articles agreeing with what has been already proved, ſhow the neceſſity of attending to the ſigns of the quantities.

PROP. XXVII.

In a right angled ſpherical triangle, rad. : tang. of the hypothenuſe :: coſ. of either angle at the hypothenuſe : tan. of the leg adjacent to that angle.

219. For (125) rad. : coſ. CEF, or coſ. A, :: $CE : EF :: \frac{CE \times AO}{OE} : \frac{EF \times AO}{OE}$ :: tan. AC : tan. AB (78), and alternando, rad. : tan. AC :: coſ. A : tan. AB. In like manner, the propoſition is proved for the angle C, and it's adjacent leg BC; ſee the obſervation to Art. 208. FIG. 24.

PROP. XXVIII.

In a right angled ſpherical triangle, rad. : coſ. of the hypothenuſe :: tan. of the angle at the baſe : cotan. of the vertical angle.

220. For (212) rad. × coſ. AC = cot. C × cot. A = cot. $C \times \frac{r^2}{\tan. A}$ (82), or tan. A × coſ. $AC = r \times$ cot. C; hence, r : coſ. AC :: tan. A : cotan. C.

PROP. XXIX.

In any ſpherical triangle, the ſines of the ſides have the ſame proportion to each other as the ſines of their oppoſite angles.

FIG. 25. 221. Let BAC be a triangle, and from any angle, as A, draw a perpendicular AD to the oppoſite ſide BC; then (212),

Rad. × ſin. AD = ſin. AB × ſin. B
Rad. × ſin. AD = ſin. AC × ſin. C;

Hence, ſin. AB : ſin. AC :: ſin. C : ſin. B*.

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* If the ſides of a triangle be decreaſed without limit, the ſines or tangents of the ſides, approach in length to the ſides themſelves as their limit, and the triangle approaches to a plane triangle as it's limit; therefore for a plane triangle, AB : AC :: ſin. C : ſin. B, as is proved in Prop. 15. Plane Trigonometry. In like manner, in all caſes of ſpherical triangles, when the ſines or tangents of the ſides enter, the ſides themſelves may be ſubſtituted, and the concluſions will be true for plane triangles.

222. Cor. Hence, ſin. AB + ſin. AC : ſin. $AB \sim$ ſin. AC :: ſin. C + ſin. B : ſin. $C \sim$ ſin. B; therefore (112) tan. $\frac{1}{2}(AB+BC)$: tan. $\frac{1}{2}(AB \sim AC)$ :: tan. $\frac{1}{2}(C+B)$: tan. $\frac{1}{2}(C \sim B)$.

223. In deducing *formula* for the ſolution of oblique angled triangles, by letting fall a perpendicular from the vertical angle upon the baſe, if we ſuppoſe the perpendicular to fall within the baſe, and the ſides and angles to be leſs than 90°, the formula which we deduce will anſwer in all other caſes, regard being had to the ſigns of the quantities (85, 86). For if the ſigns of the terms be once true, if they be made to vary according to the variation of the ſigns of the factors which compoſe them, they muſt continue true.

Prop. XXX.

In any ſpherical triangle, if a perpendicular be drawn from the vertical angle to the baſe, the coſines of the ſegments of the baſe have the ſame proportion to each other as the coſines of the adjacent ſides[*].

224. The conſtruction of the laſt Propoſition remaining, we have (212),

$$\text{Rad.} \times \text{coſ.}\, AB = \text{coſ.}\, AD \times \text{coſ.}\, BD$$
$$\text{Rad.} \times \text{coſ.}\, AC = \text{coſ.}\, AD \times \text{coſ.}\, CD$$

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[*] The vertical angle and baſe are here the repreſentatives of any angle and it's oppoſite ſide, it being arbitrary which ſide you make the baſe. And the ſame is to be underſtood, whenever the ſame terms are uſed. Alſo, when the angle A is mentioned, it means the whole angle BAC.

Hence, by multiplying croſs-ways, coſ. $AB \times$ coſ. CD $=$ coſ. $AC \times$ coſ. BD, and coſ. AB : coſ. AC :: coſ. BD : coſ. CD.

Hence, *if any two ſides AC, AB, be of the ſame, or different affections, the ſegments CD, BD of the third ſide are of the ſame, or different affections.*

225. Cor. By compoſition and diviſion, coſ. AB $+$ coſ. AC : coſ. $AB \sim$ coſ. AC :: coſ. $BD +$ coſ. CD : coſ. $BD \sim$ coſ. CD; therefore (112) cotan. $\frac{1}{2}(AB + AC)$: tan. $\frac{1}{2}(AB \sim AC)$ :: cotan. $\frac{1}{2}(BD + CD)$: tan. $\frac{1}{2}(BD \sim CD)$ :: cotan. $\frac{1}{2}BC$: tan. $\frac{1}{2}(BD \sim CD)$, or, :: cotan. $\frac{1}{2}(BD + CD)$: tan. $\frac{1}{2}BC$, according as the perpendicular falls within or without the triangle; hence, the three ſides of the triangle being given, three terms of the proportion are known, therefore we get the other term, and conſequently we know $BD - CD$, or $BD + CD$; knowing therefore the ſum and difference of BD and BC, we get each. Thus we reſolve a triangle, having the three ſides given, into two right angled triangles, in each of which we know two ſides; hence, we can find the angles (212), and therefore the angles of the given triangle are known.

Prop. XXXI.

The ſame construction remaining, the coſines of the angles at the baſe have the ſame proportion to each other as the ſines of the correſponding ſegments of the vertical angle.*

226. For (212) rad. × coſ. $B =$ coſ. $AD \times$ ſin. BAD
rad. × coſ. $C =$ coſ. $AD \times$ ſin. CAD,

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* The two parts into which the vertical angle is divided, are here called *Segments*.

Hence, by multiplying croſs-ways, coſ. $B \times$ ſin. CAD $=$ coſ. $C \times$ ſin. BAD, and coſ. C : coſ. B :: ſin. CAD : ſin. BAD.

227. Cor. 1. By compoſition and diviſion, coſ. C $+$ coſ. B : coſ. $C \sim$ coſ. B :: ſin. $CAD +$ ſin. BAD : ſin. $CAD \sim$ ſin. BAD; therefore (112) cotan. $\frac{1}{2}\,(C+B)$: tan. $\frac{1}{2}\,(C \sim B)$ :: tan. $\frac{1}{2}\,(CAD+BAD)$: tan. $\frac{1}{2}\,(CAD \sim BAD)$ :: tan. $\frac{1}{2}\,A$: tan. $\frac{1}{2}\,(CAD \sim BAD)$, or, :: tan. $\frac{1}{2}\,(CAD+BAD)$: tan. $\frac{1}{2}\,A$, according as the perpendicular falls within or without the triangle; hence, the three angles of the triangle being given, three terms of the proportion are known, therefore we get the other term, and conſequently we get $CAD \sim BAD$, or $CAD+BAD$; knowing therefore the ſum and difference of theſe two angles, we get each angle. Thus we reſolve a triangle, having the three angles given, into two right angled triangles, in each of which we know the two angles; hence, we can find the ſides (212), and therefore the ſides of the given triangle are known.

228. Cor. 2. Let the perpendicular fall within the baſe. The cotan. $\frac{1}{2}\,(C+B) = \dfrac{\text{tan. } \frac{1}{2}\,(C \sim B) \times \text{tan. } \frac{1}{2}A}{\text{tan. } \frac{1}{2}\,(CAD \sim BAD)}$, or $\dfrac{1}{\text{tan. } \frac{1}{2}\,(C+B)} = \dfrac{\text{tan. } \frac{1}{2}\,(C \sim B)}{\text{cot. } \frac{1}{2}A \times \text{tan. } \frac{1}{2}\,(CAD \sim BAD)}$, ſubſtituting $r^2 \div$ *tan.* for *cotan.* and $r^2 \div$ *cotan.* for *tan.* (82). Hence, tan. $\frac{1}{2}\,(C+B) = \dfrac{\text{cot. } \frac{1}{2}A \times \text{tan. } \frac{1}{2}\,(CAD \sim BAD)}{\text{tan. } \frac{1}{2}\,(C \sim B)}$.

Prop. XXXII.

The ſame conſtruction remaining, the coſines of the ſegments of the angle at the vertex are to each other reciprocally as the tangents of the ſides to which they are adjacent.

229. For (219) $\text{rad.}\times\text{tan.}\,AD=\text{coſ.}\,BAD\times\text{tan.}\,AB$
$\text{rad.}\times\text{tan.}\,AD=\text{coſ.}\,CAD\times\text{tan.}\,AC$;

hence, $\text{coſ.}\,BAD:\text{coſ.}\,CAD::\text{tan.}\,AC:\text{tan.}\,AB$.

230. Cor. Hence, $\text{tan.}\,AB+\text{tan.}\,AC:\text{tan.}\,AB\sim\text{tan.}\,AC::\text{coſ.}\,CAD+\text{coſ.}\,BAD:\text{coſ.}\,CAD\sim\text{coſ.}\,BAD$, or (113, 112), $\text{ſin.}\,(AB+AC):\text{ſin.}\,(AB\sim AC)::\text{cot.}\,\frac{1}{2}(CAD+BAD)$, or, $\text{cot.}\,\frac{1}{2}A:\text{tan.}\,\frac{1}{2}(CAD\sim BAD)$; but (97) $\frac{2\,\text{ſin.}\,\frac{1}{2}(AB+AC)\times\text{coſ.}\,\frac{1}{2}(AB+AC)}{r}=\text{ſin.}\,(AB+AC)$, and $\frac{2\,\text{ſin.}\,\frac{1}{2}(AB\sim AC)\times\text{coſ.}\,\frac{1}{2}(AB\sim AC)}{r}=\text{ſin.}\,(AB\sim AC)$; hence, $\text{ſin.}\,\frac{1}{2}(AB+AC)\times\text{coſ.}\,\frac{1}{2}(AB+AC):\text{ſin.}\,\frac{1}{2}(AB\sim AC)\times\text{coſ.}\,\frac{1}{2}(AB\sim AC)::\text{cot.}\,\frac{1}{2}A:\text{tan.}\,\frac{1}{2}(CAD\sim BAD)$.

Prop. XXXIII.

The ſame conſtruction remaining, the ſines of the ſegments of the baſe are to each other reciprocally as the tangents of the adjacent angles.

231. For (210) $\text{rad.}\times\text{tan.}\,AD=\text{ſin.}\,BD\times\text{tan.}\,B$,
$\text{rad.}\times\text{tan.}\,AD=\text{ſin.}\,CD\times\text{tan.}\,C$;

hence, $\text{ſin.}\,BD:\text{ſin.}\,CD::\text{tan.}\,C:\text{tan.}\,B$.

232. Preparatory to the ſolution of the next five Propoſitions, the general rule is, to let fall a perpendicular from the extremity of a given ſide, and oppoſite to a given angle.

PROP. XXXIV.

In any ſpherical triangle, given two ſides and the included angle, to find the other parts.

233. Let *BAC* be the triangle, and let there be given *AB*, *BC*, and the angle *B*. Draw *AD* perpendicular to *BC*; then,

I. $\text{Tan. } BD = \frac{r \times \text{coſ. } B}{\text{cot. } AB}$ (212).

II. $CD = CB \pm BD$.

III. $\text{Coſ. } AC = \frac{\text{coſ. } AB \times \text{coſ. } CD}{\text{coſ. } BD}$ (224).

IV. $\text{Sin. } C = \frac{\text{ſin. } AB \times \text{ſin. } B}{\text{ſin. } AC}$ (221).

V. $\text{Sin. } A = \frac{\text{ſin. } BC \times \text{ſin. } B}{\text{ſin. } AC}$ (221).

This Propoſition may be alſo thus ſolved; and here, let us ſuppoſe *AB*, *AC* and the angle *A* to be given. By Cor. Prop. 29. $\text{tan. } \frac{1}{2}(AB + AC) : \text{tan. } \frac{1}{2}(AB \sim AC) :: \text{tan. } \frac{1}{2}(C + B) : \text{tan. } \frac{1}{2}(C \sim B)$. Subſtitute for $\text{tan. } \frac{1}{2}(C + B)$ it's value found in Art. 228. and we have, $\text{tan. } \frac{1}{2}(AB + AC) : \text{tan. } \frac{1}{2}(AB \sim AC) :: \text{cot. } \frac{1}{2}A \times \text{tan. } \frac{1}{2}(CAD \sim BAD) : \overline{\text{tan. } \frac{1}{2}(C \sim B)}^2$; multiply the terms of this proportion, by the terms of the laſt proportion in Art. 230, ſubſtituting *ſin.* × *r* for *coſ.* × *tan.* and we have, $\overline{\text{ſin. } \tfrac{1}{2}(AB+AC)}^2 : \overline{\text{ſin. } \tfrac{1}{2}(AB\sim AC)}^2 :: \overline{\text{cot. } \tfrac{1}{2}A}^2 : \overline{\text{tan. } \tfrac{1}{2}(C\sim B)}^2$; hence, $\text{ſin. } \tfrac{1}{2}(AB+AC) : \text{ſin. } \tfrac{1}{2}(AB\sim AC) :: \text{cot. } \tfrac{1}{2}A : \text{tan. } \tfrac{1}{2}(C\sim B)$, and

$$\text{tan. } \tfrac{1}{2}(C\sim B) = \text{cot. } \tfrac{1}{2}A \times \frac{\text{ſin. } \tfrac{1}{2}(AB\sim AC)}{\text{ſin. } \tfrac{1}{2}(AB+AC)}.$$

Again, from the firſt of the above proportions,

$$\text{tan. } \tfrac{1}{2}(C\sim B) = \text{tan. } \tfrac{1}{2}(C+B) \times \frac{\text{tan. } \tfrac{1}{2}(AB\sim AC)}{\text{tan. } \tfrac{1}{2}(AB+AC)};$$

make this equal to the laſt value of $\text{tan. } \tfrac{1}{2}(C\sim B)$, and we get,

$$\text{tan. } \tfrac{1}{2}(C+B) = \text{cot. } \tfrac{1}{2}A \times \frac{\text{coſ. } \tfrac{1}{2}(AB\sim AC)}{\text{coſ. } \tfrac{1}{2}(AB+AC)};$$

hence, $\text{coſ. } \tfrac{1}{2}(AB+AC) : \text{coſ. } \tfrac{1}{2}(AB\sim AC) :: \text{cot. } \tfrac{1}{2}A : \text{tan. } \tfrac{1}{2}(C+B)$.

Thus we get the half ſum (S) and half difference (D) of the angles C and B; hence, $S+D$, and $S-D$ are the two angles.

Hence, $\text{ſin. } C : \text{ſin. } AB :: \text{ſin. } A : \text{ſin. } BC$ the other ſide.

234. As $\text{cot. } \tfrac{1}{2}A$, and $\text{coſ. } \tfrac{1}{2}(AB\sim AC)$ muſt be both poſitive, $\text{tan. } \tfrac{1}{2}(C+B)$, and $\text{coſ. } \tfrac{1}{2}(AB+AC)$ muſt have the ſame ſign; hence, in any ſpherical triangle, *half the ſum of any two ſides, is of the ſame affection as half the ſum of their oppoſite angles.* This conſideration will frequently take away an ambiguity which

might otherwiſe ariſe. For inſtance, if two ſides and an angle oppoſite to one of them be given, to find the angle oppoſite to the other ſide; or, if two angles and a ſide oppoſite to one of them be given, to find the ſide oppoſite to the other angle; here the required quantity, coming out in terms of a ſine, will, from that conſideration only, be ambiguous (87); but from the above property, the ambiguity will frequently be taken away. It will not always be taken away, becauſe each of the arcs, or of the angles, correſponding to the ſine of the required quantity, may be ſuch, that when added to the other arc, or angle, the half ſum may be of the ſame affection.

PROP. XXXV.

Given two ſides, and an angle oppoſite to one of them, to find the other parts.

235. Let there be given AB, BC, and the angle C. Draw BE perpendicular to AC, produced if neceſſary. Then,

I. $\text{Sin. } A = \frac{\text{ſin. } C \times \text{ſin. } BC}{\text{ſin. } AB}$ (221).

II. $\text{Tan. } AE = \frac{r \times \text{coſ. } BAE}{\text{cot. } AB}$ (212).

III. $\text{Tan. } CE = \frac{r \times \text{coſ. } C}{\text{cot. } BC}$ (212).

IV. $AC = CE \pm AE$.

V. $\text{Sin. } B = \frac{\text{ſin. } C \times \text{ſin. } AC}{\text{ſin. } AB}$ (221).

PROP.

PROP. XXXVI.

Given one ſide, and the adjacent angles, to find the other parts.

236. Let BC, and the angles B, C, be given. Then,

I. $\text{Sin. } BE = \frac{\text{ſin. } BC \times \text{ſin. } C}{r}$ (212).

II. $\text{Cot. } CBE = \frac{r \times \text{coſ. } BC}{\text{cot. } C}$ (212).

III. $\text{Sin. } BAE = \frac{r \times \text{ſin. } BE}{\text{ſin. } AB}$ (212).

IV. $\text{Sin. } AB = \frac{\text{ſin. } BC \times \text{ſin. } C}{\text{ſin. } A}$ (221).

V. $\text{Sin. } AC = \frac{\text{ſin. } BC \times \text{ſin. } B}{\text{ſin. } A}$ (221).

PROP. XXXVII.

Given one ſide, an adjacent angle, and the oppoſite angle, to find the other parts.

237. Let the ſide AB, and the angles B and C be given. Then,

I. $\text{Sin. } AC = \frac{\text{ſin. } AB \times \text{ſin. } B}{\text{ſin. } C}$ (221).

II. $\text{Tan. } BD = \frac{r \times \text{coſ. } B}{\text{cot. } AB}$ (212).

III. Tan. $CD = \frac{r \times \text{coſ.}\ C}{\text{cot.}\ AC}$ (212).

IV. $BC = CD \pm BD$.

V. Sin. $A = \frac{\text{ſin.}\ BC \times \text{ſin.}\ C}{\text{ſin.}\ AB}$ (221).

Prop. XXXVIII.

Given two angles, and the ſide included between them, to find the other parts.

238. Let B, BAC and AB be given, Then,

I. Cot. $BAD = \frac{r \times \text{coſ.}\ BA}{\text{cot.}\ B}$ (212).

II. $CAD = A \pm BAD$.

III. Coſ. $C = \frac{\text{coſ.}\ B \times \text{ſin.}\ CAD}{\text{ſin.}\ BAD}$ (226).

IV. Sin. $AC = \frac{\text{ſin.}\ AB \times \text{ſin.}\ B}{\text{ſin.}\ C}$ (221).

V. Sin. $BC = \frac{\text{ſin.}\ AB \times \text{ſin.}\ A}{\text{ſin.}\ C}$ (221).

This Propoſition may alſo be thus ſolved. In the ſupplemental triangle, we have given PR, QR and the included angle R. Hence, by Prop. 34. FIG. 19.

Sin. $\frac{1}{2}(PR+QR)$: ſin. $\frac{1}{2}(PR-QR)$ :: cot. $\frac{1}{2}R$: tan. $\frac{1}{2}(P-Q)$

Coſ. $\frac{1}{2}(PR+QR)$: coſ. $\frac{1}{2}(PR-QR)$ :: cot. $\frac{1}{2}R$: tan. $\frac{1}{2}(P+Q)$.

Now $B = 180° - PR$, $A = 180° - QR$, $AB = 180° - R$, $BC = 180° - P$, $AC = 180° - Q$; hence, ſin. or coſ.

$\frac{1}{2}(A+B) =$ ſin. or coſ. $\frac{1}{2}(360° - \overline{PR+QR}) =$ ſin. or coſ. $\frac{1}{2}(PR+QR)$; ſin. or coſ. $\frac{1}{2}(A-B) =$ ſin. or coſ. $\frac{1}{2}(PR-QR)$; tan. $\frac{1}{2}AB =$ tan. $\frac{1}{2}(180° - R) =$ tan. $(90° - \frac{1}{2}R) =$ cot. $\frac{1}{2}R$; tan. $\frac{1}{2}(AC-BC) =$ tan. $\frac{1}{2}(P-Q)$; and tan. $\frac{1}{2}(AC+BC) =$ tan. $\frac{1}{2}(360° - \overline{P+Q}) =$ tan. $\frac{1}{2}(P+Q)$; hence, by ſubſtitution,

Sin. $\frac{1}{2}(A+B)$: ſin. $\frac{1}{2}(A-B)$ :: tan. $\frac{1}{2}AB$: tan. $\frac{1}{2}(AC-BC)$,
Coſ. $\frac{1}{2}(A+B)$: coſ. $\frac{1}{2}(A-B)$ :: tan. $\frac{1}{2}AB$: tan. $\frac{1}{2}(AC+BC)$.

Knowing therefore (S) half the ſum of AC and BC, and (D) half the difference of AC and BC, we have, $S+D =$ the greater ſide, and $S-D =$ the leſs ſide.

Prop. XXXIX.

In any ſpherical triangle, as the rectangle under the ſines of any two ſides : the ſine of $\frac{1}{2}$ *(ſum of the ſame two ſides + the third ſide)* × *the ſine of* $\frac{1}{2}$ *(ſum of the ſame two ſides − the third ſide) :: the ſquare of radius : the ſquare of the coſine of half the angle comprehended between the before-mentioned two ſides.*

FIG. 26. 239. Let ABC be the given triangle, O the center of the ſphere, and draw the radii AO, BO; and let ECP be a ſmall circle perpendicular to AO, and EHP a diameter of it; then (166) $AC = AE = AP$; let alſo LCD be a ſmall circle perpendicular to BO, and $LmFD$ a diameter of it, interſecting EHP in m, and join Cm; then as the plane of each circle ECP, LCD, is perpendicular to the circle $AEDB$, their common interſection Cm is perpendicular to the ſame circle (Euc. B. XI. p. 19.); therefore Cm is perpendicular to EHP; alſo, $BC=BL=BD$ (166); join ED, CH, LO, EO, and draw Os perpendicular to ED. Now to the radius EH, Em is the verſed-ſine of the angle EHC, or of EAC (173); hence,

$EH : Em ::$ rad. : ver.-ſin. EAC
$Em : DE ::$ ſin. EDm : ſin. EmD, or ſin. DmP (128)

$\therefore EH : DE ::$ rad.×ſin.EDm : ver.-ſin.EAC×ſin.DmP

But $EH=$ ſin. $AE=$ ſin. AC, to the radius (r) of the ſphere; alſo, $ED = 2Es =$ (90) 2 ſin. $\frac{1}{2}\,EABD =$ 2 ſin. $\frac{1}{2}\,(EA+AB+BD) =$ 2 ſin. $\frac{1}{2}\,(AC+AB+BC)$; and ſin. EDm, or EDL, or ſin. $\frac{1}{2}\,EOL$ (Euc. B. III. p. 20.) or, ſin. $\frac{1}{2}\,EL =$ ſin. $\frac{1}{2}\,(AE-AL) =$ ſin. $\frac{1}{2}\,(AC+AB-BL) =$ ſin. $\frac{1}{2}\,(AC+AB-BC)$. Again, the angles OvF, mvH, are the complements of AOB, DmP reſpectively, and as the two former are equal, we have $DmP=AOB$; therefore ſin. $DmP=$ ſin. $AOB=$ ſin. AB. Hence, the above proportion becomes, ſin. AC : 2 ſin. $\frac{1}{2}\,(AC+AB+BC)$ :: r × ſin. $\frac{1}{2}\,(AC+AB-BC)$: ver.-ſin. EAC × ſin. AB, or*, ſin. AC × ſin. AB : ſin. $\frac{1}{2}\,(AC+AB+BC)$ × ſin.

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* That the next proportion follows from the preceding, is manifeſt from this, that in any proportion, a factor may be transferred from one extreme term to the other, or from one of the mean terms to the other; or one of the mean and one of the extreme terms may be multiplied by the ſame quantity, and the product of the extremes and means will ſtill continue equal, and therefore the proportion will be true. In the caſe above, ſin. AB is transferred from the laſt to the firſt term; ſin. $\frac{1}{2}\,(AC+AB-BC)$ is transferred from the third to the ſecond term; the ſecond and fourth terms are multiplied by $\frac{1}{2}$, and the third and fourth by r.

$\times$ ſin. $\frac{1}{2}(AC+AB-BC)$ :: r^2 : $\frac{1}{2}r\times$ ver.-ſin. EAC, or, $\overline{\text{coſ.}\ \frac{1}{2}CAB}^2$ (95).

The three ſides of the triangle being given, the computation of the angle CAB immediately follows: for coſ. $\frac{1}{2}$ CAB

$$=\sqrt{\frac{r^2\times \text{ſin.}\ \frac{1}{2}(AC+AB+BC)\times\text{ſin.}\ \frac{1}{2}(AC+AB-BC)}{\text{ſin.}\ AC\times\text{ſin.}\ AB}},$$

an expreſſion ſimilar to that for plane triangles, Art. 134. Hence, log. coſ. $\frac{1}{2}$ CAB = $\frac{1}{2}$ (log. ſin. $\frac{1}{2}(AC+AB+BC)$ + log. ſin. $\frac{1}{2}(AC+AB-BC)$ + ar. co. log. ſin. AC + ar. co. log. ſin. AB). Here can be no ambiguity, as $\frac{1}{2}$ CAB muſt be leſs than 90°.

This rule (as Dr. MASKELYNE obſerves) is not proper to be uſed when A is near 0°, becauſe the difference of the log. coſines of $\frac{1}{2}A$ for 1″ is then very ſmall; and *very* near 0°, the log. coſines of $\frac{1}{2}A$ are the ſame for an arc of many ſeconds, owing to the logarithms being continued only to ſeven places; in this caſe therefore it will be proper to uſe the next rule. This rule is moſt proper to be uſed, when A is between 90° and 180°; for then there is a conſiderable difference of the log. coſines of $\frac{1}{2}A$ for 1″.

PROP.

PROP. XL.

As the rectangle under the ſines of any two ſides : ſin. ½ (third ſide + difference of the other two ſides) × ſin. ½ (third ſide − difference of the other two ſides) :: the ſquare of radius : the ſquare of the ſine of half the angle included by the before-mentioned two ſides.

240. For $r^2 - \overline{\text{ſin. } \tfrac{1}{2}\, CAB}^2 = \overline{\text{coſ. } \tfrac{1}{2}\, CAB}^2 = \dfrac{r^2 \times \text{ſin. } \tfrac{1}{2}\,(AC+AB+BC) \times \text{ſin. } \tfrac{1}{2}\,(AC+AB-BC)}{\text{ſin. } AC \times \text{ſin. } AB}$; hence, $\text{ſin. } \tfrac{1}{2}\, CAB =$

$$\sqrt{\frac{r^2 \times \text{ſin.} AC \times \text{ſin.} AB - r^2 \times \text{ſin.} \tfrac{1}{2}(AC+AB+BC) \times \text{ſin.} \tfrac{1}{2}(AC+AB-BC)}{\text{ſin. } AC \times \text{ſin. } AB}};$$

but (107) $\text{ſin. } AC \times \text{ſin. } AB = r \times \tfrac{1}{2} \text{ coſ. } (AC-AB) - r \times \tfrac{1}{2} \text{ coſ. } (AC+AB)$; and $\text{ſin. } \tfrac{1}{2}\,(AC+AB+BC) \times \text{ſin. } \tfrac{1}{2}\,(AC+AB-BC) = r \times \tfrac{1}{2} \text{ coſ. } BC - r \times \tfrac{1}{2} \text{ coſ. } (AC+AB)$; hence, omitting r^2, the numerator becomes $r \times \tfrac{1}{2} \text{ coſ. } (AC-AB) - r \times \tfrac{1}{2} \text{ coſ. } BC =$ (111) $\text{ſin. } \tfrac{1}{2}\,(BC+AC-AB) \times \text{ſin. } \tfrac{1}{2}\,(BC+AB-AC) =$ (if $D = AC - AB$) $\text{ſin. } \tfrac{1}{2}\,(BC+D) \times \text{ſin. } \tfrac{1}{2}\,(BC-D)$. Hence,

$$\text{ſin. } \tfrac{1}{2}\, CAB = \sqrt{\frac{r^2 \times \text{ſin. } \tfrac{1}{2}\,(BC+D) \times \text{ſin. } \tfrac{1}{2}\,(BC-D)}{\text{ſin. } AC \times \text{ſin. } AB}},$$

and this equation gives the following analogy; $\text{ſin. } AC \times \text{ſin. } AB : \text{ſin. } \tfrac{1}{2}\,(BC+D) \times \text{ſin. } \tfrac{1}{2}\,(BC-D) :: r^2 : \overline{\text{ſin. } \tfrac{1}{2}\, CAB}^2$.

From the equation we have, $\text{log. ſin. } \tfrac{1}{2}\, CAB = \tfrac{1}{2}\Big(\text{log. ſin. } \tfrac{1}{2}\,(BC+D) + \text{log. ſin. } \tfrac{1}{2}\,(BC-D) + \text{ar. co. log. ſin. } AC + \text{ar. co. log. ſin. } AB\Big)$.

This rule is not proper, to be uſed when A is near 180°, becauſe the difference of the log. ſines of $\frac{1}{2}A$ for 1″ is then very ſmall; and *very* near 180°, the log. ſines of $\frac{1}{2}A$ are the ſame for an arc of many ſeconds; in this caſe therefore it will be proper to uſe the preceding rule. This rule is moſt proper to be uſed, when A is leſs than 90°; for then there is a conſiderable difference of the log. ſines of $\frac{1}{2}A$ for 1″.

PROP. XLI.

In any ſpherical triangle, as the rectangle under the ſines of any two angles : coſ. of ½ (third angle + difference of the other two angles) × coſ. of ½ (third angle − difference of the other two angles) :: the ſquare of radius : the ſquare of the coſine of half the ſide comprehended between the before-mentioned two angles.

FIG. 19. 241. Let PQR be the ſupplemental triangle to ABC; then (191) $180° - Q \doteq AC$, $180° - R = AB$, $180° - P = BC$; and putting $D = AB - AC$, as in Prop. 40, we have, $\frac{1}{2}(BC + D) = 90° - \frac{1}{2}(P + R - Q)$, and $\frac{1}{2}(BC - D) = 90° - \frac{1}{2}(P + Q - R)$; hence (66), ſin. $\frac{1}{2}(BC + D) =$ coſ. $\frac{1}{2}(P + R - Q)$, and ſin. $\frac{1}{2}(BC - D) =$ coſ. $\frac{1}{2}(P + Q - R)$; alſo, ſin. $AC \doteq$ ſin. Q, and ſin. $AB =$ ſin. R (63); laſtly, ſin. $\frac{1}{2}CAB$ = ſin. $\frac{1}{2}$ ſup. QR (191) = coſ. $\frac{1}{2}QR$ (66), becauſe half the ſupplement of an arc = complement of half that arc. Subſtitute theſe values in the proportion in Prop. 40, and we have, ſin. $Q \times$ ſin. R : coſ. $\frac{1}{2}(P + R - Q) \times$ coſ. $\frac{1}{2}(P + Q - R)$ :: $\overline{\text{rad.}}^2 : \overline{\text{coſ. } \tfrac{1}{2} QR}^2$.

This rule is not to be uſed when QR is near 0°, but when it is between 90° and 180°; ſee Prop. 39.

PROP. XLII.

In any ſpherical triangle, as the rectangle under the ſines of any two angles : coſ. $\frac{1}{2}$ *(ſum of the ſame two angles + third angle)* × *coſ.* $\frac{1}{2}$ *(ſum of the ſame two angles − third angle) :: the ſquare of radius : the ſquare of the ſine of half the ſide comprehended between the before-mentioned two angles.*

242. For $AC+AB+BC=3\times 180^\circ-(P+Q+R)$, therefore ſin. $\frac{1}{2}(AC+AB+BC)=$ ſin. $\left(270^\circ-(Q+R+P)\right)=$ coſ. $\frac{1}{2}(Q+R+P)$; and $AC+AB-BC=180^\circ+P-Q-R$, therefore ſin. $\frac{1}{2}(AC+AB-CB)=$ ſin. $\left(90^\circ-\frac{1}{2}(Q+R-P)\right)=$ coſ. $\frac{1}{2}(Q+R-P)$; alſo, ſin. $AC=$ ſin. Q, ſin. $AB=$ ſin. R, and coſ. $\frac{1}{2}A=$ coſ. $\frac{1}{2}$ ſup. $QR=$ ſin. $\frac{1}{2}QR$. Subſtitute theſe values in the proportion in Prop. 40. and we have, ſin. $Q\times$ ſin. R : coſ. $\frac{1}{2}(Q+R+P)\times$ coſ. $\frac{1}{2}(Q+R-P)$ :: $\overline{\text{rad.}}^2$: $\overline{\text{ſin. }\frac{1}{2}QR}^2$.

This rule is not proper to be uſed, when QR is near 180°, but is moſt proper when QR is leſs than 90°; ſee Prop. 40.

Thus the three angles of a triangle being given, the ſides may be found. Theſe data are therefore ſufficient to determine the triangle.

What we have hitherto given, is ſufficient for the ſolution of all the caſes of ſpherical triangles; but as there are ſome theorems which are very uſeful in aſtronomical and nautical enquiries, we ſhall here give them; and they are alſo further valuable, as they ſhow the affections of the ſides and angles, and thereby frequently ſerve to take away the ambiguity which ſometimes ariſes in the numerical ſolution of triangles.

Prop. XLIII.

In any ſpherical triangle ABC, the coſine of B = $\frac{r^2 \times \text{coſ. } AC - r \times \text{coſ. } AB \times \text{coſ. } BC}{\text{ſin. } AB \times \text{ſin. } BC}$.

FIG. 25. 243. By Art. 224. coſ. AB : coſ. AC :: coſ. BD : coſ. CD :: coſ. BD : coſ. $(BC - BD)$ :: coſ. BD : $\frac{\text{coſ. } BC \times \text{coſ. } BD + \text{ſin. } BC \times \text{ſin. } BD}{r}$ (103) :: $\left(\text{dividing by coſ. } BD\text{, and ſubſtituting } \frac{\text{tan. } BD}{r} \text{ for } \frac{\text{ſin. } BD}{\text{coſ. } BD}\right)$ 1 : $\frac{r \times \text{coſ. } BC + \text{ſin. } BC \times \text{tan. } BD}{r^2}$ :: $\left(\text{by ſubſtituting } \frac{\text{coſ. } B \times \text{tan. } AB}{r} \text{ for tan. } BD \text{ (219)}\right)$ 1 : $\frac{r^2 \times \text{coſ. } BC + \text{ſin. } BC \times \text{coſ. } B \times \text{tan. } AB}{r^3}$; hence,

$$\text{coſ. } B = \frac{r^3 \times \text{coſ. } AC - r^2 \times \text{coſ. } AB \times \text{coſ. } BC}{\text{coſ. } AB \times \text{tan. } AB \times \text{ſin. } BC} = \left(\text{as coſ. } AB \times \text{tan. } AB = r \times \text{ſin. } AB \text{ (78)}\right) \frac{r^2 \times \text{coſ. } AC - r \times \text{coſ. } AB \times \text{coſ. } BC}{\text{ſin. } AB \times \text{ſin. } BC}.$$

The coſines of each of the other angles will be expreſſed by the ſame form, taking, in each caſe, the ſides which are ſimilarly ſituated in reſpect to the angle whoſe value is expreſſed.

244. As the denominator is always poſitive, the angle B will be leſs or greater than 90°, according as the numerator is poſitive or negative. Hence, we immediately diſcover the following affections.

If the ſide oppoſite to B be leſs than 90°, and one of the other two ſides be greater and one leſs than 90°, B is leſs than 90°.

If the ſide oppoſite to B be greater than 90°, and each of the other two ſides be leſs than 90°, B is greater than 90°.

If the ſide oppoſite to B be greater than 90°, and each of the other two ſides be alſo greater than 90°, B is greater than 90°. Hence, when the three ſides are each greater than 90°, each angle is greater than 90°.

The angle B is of the ſame affection as the oppoſite ſide AC, when one adjacent ſide, as AB, is either $=AC$, or $=180° - AC$, or between AC and $180° - AC$; for in each caſe, coſ. AC is either equal to, or greater than coſ. AB; and therefore coſ. AC is greater than coſ. $AB \times$ coſ. BC.

245. The expreſſion for coſ. B, as it ſtands here, is not adapted to a logarithmic computation, but it may be eaſily reduced to it thus. The coſ. $B =$

$$r^2 \times \frac{\text{coſ.}\,AC - \dfrac{\text{coſ.}\,AB \times \text{coſ.}\,BC}{r}}{\text{ſin.}\,AB \times \text{ſin.}\,BC};$$

let M be ſuch an arc, that coſ. $M = \frac{\text{coſ.}\,AB \times \text{coſ.}\,BC}{r}$; then, coſ. B

The above expreſſion may be reduced to a logarithmic calculation, by a proceſs ſimilar to that of the laſt Propoſition. For put coſ. $M = \frac{\text{coſ. } B \times \text{coſ. } A}{r}$; then, coſ. $AB = r^2 \times \frac{\text{coſ. } C + \text{coſ. } M}{\text{ſin. } B \times \text{ſin. } A} =$ (110) $r \times \frac{2\,\text{coſ. } \frac{1}{2}(M+C) \times \text{coſ. } \frac{1}{2}(M-C)}{\text{ſin. } B \times \text{ſin. } A}$. Hence, log. coſ. $AB = 0{,}3010300 +$ log. coſ. $\frac{1}{2}(M+C) +$ log. coſ. $\frac{1}{2}(M-C) +$ ar. co. log. ſin. $B +$ ar. co. log. ſin. $A - 10$,.

PROP. XLV.

In any ſpherical triangle ABC, the tangent of $B = \frac{r^2 \times \text{ſin. } C}{\text{ſin. } BC \times \text{cot. } AC - \text{coſ. } BC \times \text{coſ. } C}$.

248. By Art. 231. tan. B : tan. C :: ſin. CD : ſin. BD :: ſin. CD : ſin. $(BC - CD)$ :: ſin. CD : $\frac{\text{ſin. } BC \times \text{coſ. } CD - \text{coſ. } BC \times \text{ſin. } CD}{r}$ (101) :: 1 : $\frac{\text{ſin. } BC \times \text{coſ. } CD}{r \times \text{ſin. } CD} - \frac{\text{coſ. } BC}{r}$ :: 1 : $\frac{\text{ſin. } BC \times \text{cot. } CD}{r^2} - \frac{\text{coſ. } BC}{r}$; but (212) $r \times$ coſ. $C =$ tan. $CD \times$ cot. $AC = \frac{r^2}{\text{cot. } CD} \times$ cot. AC (82); therefore $\frac{\text{cot. } CD}{r} = \frac{\text{cot. } AC}{\text{coſ. } C}$; hence, tan. B : tan. C :: 1 :

ſin.

$\frac{\text{ſin. } BC \times \text{cot. } AC}{r \times \text{coſ. } C} - \frac{\text{coſ. } BC}{r}$; hence, tan. $B = \frac{r^2 \times \text{ſin. } C}{\text{ſin. } BC \times \text{cot. } AC - \text{coſ. } BC \times \text{coſ. } C}$.

The tangents of the other angles will be expreſſed by the ſame form, taking, in each caſe, the ſides and angle which are ſimilarly ſituated in reſpect to the angle whoſe value is expreſſed.

249. As the numerator is always poſitive, B will be greater or leſs than 90°, according as the denominator is negative or poſitive. Hence,

If AC and C be both leſs than 90°, and BC be greater than 90°, B is leſs than 90°.

If AC and BC be both leſs than 90°, and C be greater than 90°, B is leſs than 90°.

If BC and C be both leſs or both greater than 90°, and AC greater than 90°, B is greater than 90°.

250. To adapt this to a logarithmic computation, we have, tan. $B = \frac{r \times \text{ſin. } C}{\frac{\text{ſin. } BC \times \text{cot. } AC}{r} - \frac{\text{coſ. } BC \times \text{coſ. } C}{r}}$;

let M = an arc, ſuch that coſ. $M = \frac{\text{ſin. } BC \times \text{cot. } AC}{r}$,

and N = an arc, ſuch that coſ. $N = \frac{\text{coſ. } BC \times \text{coſ. } C}{r}$;

then, tan. $B = \frac{r \times \text{ſin. } C}{\text{coſ. } M - \text{coſ. } N} =$ (111) $\frac{r^2 \times \text{ſin. } C}{2 \text{ ſin. } \frac{1}{2}(N+M) \times \text{ſin. } \frac{1}{2}(N-M)}$. But it is manifeſt that this method is not general; for if $\frac{\text{ſin. } BC \times \text{cot. } AC}{r}$ be greater than radius, we can take no arc whoſe coſine is equal to this quantity. This method of computation therefore, which is given by M. MAUDUIT without any reſtriction, cannot always be applied. In the caſes where it is applicable, we have, log. tan. $B = 20, +$ log. ſin. $C +$ ar. co. log. $2 +$ ar. co. log. ſin. $\frac{1}{2}(N+M)$ + ar. co. log. ſin. $\frac{1}{2}(N-M) - 30, = 9{,}6989700 +$ log. ſin. $C +$ ar. co. log. ſin. $\frac{1}{2}(N+M)$ + ar. co. log. ſin. $\frac{1}{2}(N-M)$ $- 10,$,

PROP. XLVI.

In any ſpherical triangle ABC, the tangent of AB = $\frac{r^2 \times \text{ſin. } AC}{\text{ſin. } A \times \text{cot. } C + \text{coſ. } A \times \text{coſ. } AC}$.

251. By Art. 229. tan. AB : tan. AC :: coſ. CAD : coſ. BAD :: coſ. CAD : coſ. $(A - CAD)$ :: coſ. CAB : $\frac{\text{coſ. } A \times \text{coſ. } CAD + \text{ſin. } A \times \text{ſin. } CAD}{r}$ (103) :: 1 : $\frac{\text{coſ. } A}{r} + \frac{\text{ſin. } A \times \text{tan. } CAD}{r^2}$; but (212), $r \times$ coſ. $AC =$ cot. $C \times$ cot. $CAD =$ cot. $C \times \frac{r^2}{\text{tan. } CAD}$ (82); hence, $\frac{\text{tan. } CAD}{r} = \frac{\text{cot. } C}{\text{coſ. } AC}$; therefore tan. AB : tan. AC :: 1 : $\frac{\text{coſ. } A}{r} + \frac{\text{ſin. } A \times \text{cot. } C}{r \times \text{coſ. } AC}$; hence,

$$\text{tan. } AB = \frac{r \times \text{coſ. } AC \times \text{tan. } AC}{\text{ſin. } A \times \text{cot. } C + \text{coſ. } A \times \text{coſ. } AC} = \frac{r^2 \times \text{ſin. } AC}{\text{ſin. } A \times \text{cot. } C + \text{coſ. } A \times \text{coſ. } AC}.$$

The tangents of the other ſides will be expreſſed by a ſimilar form, taking, in each caſe, the angles and ſide which are ſimilarly ſituated in reſpect to the ſide whoſe value is expreſſed.

252. As the numerator is always poſitive, *AB* will be greater or leſs than 90°, according as the denominator is negative or poſitive. Hence,

If *A*, *C*, and *AC* be each leſs than 90°, *AB* is leſs than 90°.

If *C* be leſs than 90°, and *A* and *AC* be both greater than 90°, *AB* is leſs than 90°.

If *C* and *A* be both greater than 90°, and *AC* leſs than 90°, *AB* is greater than 90°.

If *C* and *AC* be both greater than 90°, and *A* be leſs than 90°, *AB* is greater than 90°.

253. To adapt this to a logarithmic computation, we have, $\text{tan. } AB = \dfrac{r \times \text{ſin. } AC}{\dfrac{\text{ſin. } A \times \text{cot. } C}{r} + \dfrac{\text{coſ. } A \times \text{coſ. } AC}{r}}$;

let M = an arc, ſuch that $\text{ſin. } M = \dfrac{\text{ſin. } A \times \text{cot. } C}{r}$,

and N = an arc, ſuch that $\text{ſin. } N = \dfrac{\text{coſ. } A \times \text{coſ. } AC}{r}$;

then, $\text{tan. } AB = \dfrac{r \times \text{ſin. } AC}{\text{ſin. } M + \text{ſin. } N} =$ (108)

$r^2 \times$

$\frac{r^2 \times \text{ſin.}\ AC}{2 \times \text{ſin.}\ \frac{1}{2}(M+N) \times \text{ſin.}\ \frac{1}{2}(M-N)}$. But this computation cannot be applied when $\frac{\text{ſin.}\ A \times \text{cot.}\ C}{r}$ is greater than radius. In caſes where it is applicable, we have, log. tan. $AB = 9{,}6989700 +$ log. ſin. $AC +$ ar. co. log. ſin. $\frac{1}{2}(M+N) +$ ar. co. log. ſin. $\frac{1}{2}(M-N) - 10$,.

THE RESOLUTION OF SPHERICAL TRIANGLES, BY LOGARITHMS.

Ex. 1. *Given the obliquity of the ecliptic* =23°. 28′, *and the right aſcenſion of the ſun* =49°. 17′; *to find it's declination, and longitude.*

FIG. 23. Let AB repreſent the equator, AC the ecliptic, A the firſt point of ♈, and BC be perpendicular to BA; then $AB = 49°. 17′$, and the angle $A = 23°. 28′$; to find the declination BC, and longitude AC.

By Art. 212. rad. × ſin. $AB =$ tan. $BC \times$ cot. A; therefore (213) 10,+log. ſin. $AB -$ log. cot. $A =$ log. tan. BC; hence,

10,+ſin. 49°. 17′. 0″ - - - - - -	19,8796375
cot. 23. 28. 0 - - - - - -	10,3623894
tan. 18. 12. 48 the *Declination* -	9,5172481

Alſo (212), rad. × coſ. A = tan. AB × cot. AC; therefore (213) 10,+log. coſ. A − log. tan. AB = log. cot. AC; hence,

10,+coſ. 23°. 28′. 0″ - - - - - - 19,9625076
tan. 49. 17. 0 - - - - - - 10,0651775

cot. 51. 42. 37 the *Longitude* - - 9,8973301

If AB repreſent the ecliptic, AC the orbit of a planet, and C the place of a planet; then if we know the longitude of the planet upon the ecliptic, or of the point B, and the longitude of the node A, we know AB; and let BC, the heliocentric latitude of the planet, be alſo given; then (212) rad. × ſin. AB = cot. A × tan. BC; hence, tan. BC : rad. :: ſin. AB : cot. A the inclination of the orbit of the planet to the ecliptic.

The difference between the hypothenuſe and baſe of a right angled ſpherical triangle, is called the *Reduction*, becauſe you may reduce one to the other by means of their difference. And in Aſtronomical Tables, there is found, for inſtance, a Table, called *The Reduction of the Equator to the Ecliptic*, in which you have the difference between the equator and the the ecliptic, for every 10 minutes of right aſcenſion, which applied, according to the ſign, to the right aſcenſion, gives the correſponding longitude. In like manner, a Table is conſtructed to give the right aſcenſion from the longitude. By a like Table, the place of a planet in it's orbit may be reduced to the ecliptic, or the contrary.

Ex. 2. *Given the ſun's altitude at ſix o'clock* = 17°. 44′, *and it's declination* = 21°. 17′ N.; *to find it's azimuth, and the latitude of the place.*

FIG. 27. Let Z be the zenith of the place, P the pole of the equator, S the place of the ſun; then PZ being the meridian, and the ſun being ſix hours from it, the angle $ZPS = 90°$*; alſo, the angle SZP = the azimuth from the north, ZP = the complement of the latitude, PS = the complement of the declination = 68°. 43′, and ZS = the complement of the ſun's altitude = 72°. 16′. Now (212) rad. × coſ. ZS = coſ. PZ × coſ. PS; therefore (213) 10, + log. coſ. ZS − log. coſ. PS = log. coſ. PZ = log. ſin. lat. Hence,

10, + coſ. 72°. 16′. 0″ - - - - - -	19,4837117
coſ. 68 43. 0 - - - - - -	9,5598829
ſin. 57. 2. 54 the *Latitude* - -	9,9238288

Alſo (212), rad. × ſin. PS = ſin. SZ × ſin. Z; therefore (213) 10, + log. ſin. PS − log. ſin. SZ = log. ſin. Z; hence,

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* The angle ZPS here repreſents the time from *apparent* noon, or from the time when the ſun comes to the meridian, and not from 12 o'clock; this is called the *hour-angle*; and the time is found from the angle, or the angle from the time, by reducing it at the rate of 15° for 1 hour.

10, + ſin. 68°. 43′. 0″ - - - - - -	19,9693212
ſin. 72. 16. 0 - - - - - -	9,9788579
ſin. 78. 2. 12 = z the *Azimuth* -	9,9904633

Ex. 3. *Given the latitude of the place* = 52°. 12′ N. *and ſun's declination* 23°. 28′ N.; *to find it's altitude at ſix o'clock.*

Here P is a right angle, and (212) rad. × coſ. ZS = coſ. PZ × coſ. PS, therefore (213) log. coſ. PZ + log. coſ. $PS - 10$, = log. coſ. ZS = log. ſin. alt.; hence,

coſ. 37°. 48′. 0″ - - - - - - - - -	9,8977123
coſ. 66. 32. 0 - - - - - - - - -	9,6001181
ſin. 18. 20. 23 the *Altitude* - - -	9,4978304

The operation may be performed by the natural coſines; for making radius unity, we have, ſin. *Alt.* = coſ. PZ × coſ. PS = (106) ½ coſ. $(PS + PZ)$ + ½ coſ. $(PS - PZ)$ = ½ coſ. 104°. 20′ + ½ coſ. 28°. 44′ = − 0,1237813 + 0,4384333 = 0,314652 the natural ſine of 18°. 20′. 23″.

In like manner, any quantity which is compoſed of the product of two ſines, two coſines, or ſine and coſine, may be computed by Art. 104, 105, 106, 107.

Ex. 4. *Given the latitude of the place* = 52°. 12′ N. *and declination of the ſun* 23°. 28′ N.; *to find at what time the ſun comes to the prime vertical, and it's altitude at that time.*

When the ſun comes to the prime vertical, it's azimuth = 90°; therefore the angle $SZP = 90°$; alſo, $SP = 66°. 32'$ the complement of declination, and $ZP = 37°. 48'$ the complement of latitude. Now (212) rad. × coſ. ZPS = tan. ZP × cot. SP; therefore (213) log. tan. ZP + log. cot. SP − 10, = log. coſ. ZPS; hence,

tan. 37°. 48′. 0″ - - - - - -	9,8896823
cot. 66. 32. 0 - - - - - -	9,6376106
coſ. 70. 19. 18 = *ZPS* - - - -	9,5272929

And 70°. 19′. 18″ converted into time, gives 4*h*. 41′. 17″ the time from apparent noon.

Alſo (212), rad. × coſ. PS = coſ. PZ × coſ. ZS; therefore (213) 10, + log. coſ. PS − log. coſ. PZ = log. coſ. SZ = log. ſin. of the altitude; hence,

10, + coſ. 66°. 32′. 0″ - - - - - -	19,6001181
coſ. 37. 48. 0 - - - - - -	9,8977123
ſin. 30. 15. 47 the *Altitude* - -	9,7024058

Ex. 5. *Given the latitude of the place* 52°. 12′ N. *and ſun's declination* 23°. 28′ N.; *to find the time of it's riſing, and azimuth at that time.*

In this caſe, $ZS = 90°$, and it becomes a quadrantal triangle. Now (215) rad. × coſ. $P =$ cot. PZ × cot. PS; therefore log. cot. PZ + log. cot. $PS - 10$, = log. coſ. P; hence,

cot. 37°. 48′. 0″ - - - - - - 10,1103177
cot. 66. 32. 0 - - - - - - 9,6376106

coſ. 124. 1. 58 = P - - - - 9,7479283

The angle in the Tables correſponding to the log. coſ. here found, is 55°. 58′. 2″, but we here take it's ſupplement; for as PZ, PS, are of the ſame affection, the angle P muſt be greater than 90°, according to the fifth rule in Art. 216. And 124°. 1′. 58″ converted into time, gives 8*h*. 16′. 8″ the time from apparent noon at which the ſun riſes, or it riſes at 3*h*. 43′. 52″ apparent time.

Alſo (215), rad. × coſ. PS = ſin. PZ × coſ. Z; therefore 10, + log. coſ. PS − log. coſ. PZ = log. coſ. Z; hence,

10, + coſ. 66°. 32′. 0″ - - - - - - 19,6001181
coſ. 37. 48. 0 - - - - - - 9,8977123

coſ. 59. 4. 13 = Z the *azimuth* - 9,7024058

Ex. 6. *In the oblique angled triangle* ABC, *given* $AB=37^\circ.\ 10'$, $BC=68^\circ.\ 46'.\ 2''$, *and the included angle* $B=39^\circ.\ 23'$; *to find the other parts.*

By Prop. 34. we have the theorems for this computation.

I. 10, + log. coſ. B − log. cot. AB = (as log. tan. + log. cot. = 20,) log. tan. AB + log. coſ. B − 10, = log. tan. BD; hence,

tan. 37°. 10′. 0″ - - - - - - - - -	9,8797407
coſ. 39. 23. 0 - - - - - - - -	9,8881335
tan. 30. 22. 8 = BD - - - - -	9,7678742

As coſ. B, and tan. BA are both poſitive, tan. BD is poſitive; therefore there is no ambiguity.

II. Hence, $CD=68^\circ.\ 46'.\ 2''-30^\circ.\ 22'.\ 8''=38^\circ.\ 23'.\ 54''$, and the perpendicular falls within the baſe.

III. Ar. co. log. coſ. BD + log. coſ. AB + log. coſ. CD − 10, = log. coſ. AC; hence,

Ar. co. coſ. 30°. 22′. 8″ - - - - - - -	0,0640957
coſ. 37. 10. 0 - - - - - - -	9,9013938
coſ. 38. 23. 54 - - - - - -	9,8941562
coſ. 43. 37. 38 = AC - - -	9,8596457

IV. Ar. co. log ſin. AC + log. ſin. AB, + log. ſin. $B - 10$, = log. ſin. C; hence,

Ar. co. ſin.	43°. 37'. 38''	0,1611739
ſin.	37. 10 0	9,7811344
ſin.	39. 23. 0	9,8024355
ſin.	33. 45. 2 = C	9,7447438

The angle C is not ambiguous, being of the ſame affection as the angle B, becauſe the perpendicular falls within the baſe (205).

V. Ar. co. log. ſin. AC + log. ſin. $B - 10$, = log. ſin. A; hence,

Ar. co. ſin.	42°. 37'. 38''	0,1611739
ſin.	68. 46. 2	9,9694703
ſin.	39. 23. 0	9,8024355
ſin.	120. 59. 49 = A	9,9330797

The ſine of A gives A either 59°. 0'. 11'', or it's ſupplement 120°. 59'. 49''; but we muſt take the latter, otherwiſe the ſum of the three angles would be leſs than two right angles, which cannot be (192).

By this example, we may calculate the diſtance of two places on the earth's ſurface, having given their latitudes and longitudes. For if B be the pole of the earth, A, C, the two places, then we know AB, *CB*, the complements of the latitudes, and the angle *B*, the difference of the longitudes. Or, if *B* be the pole of the equator in the heavens, *A*, *C*, two ſtars, whoſe declinations and right aſcenſions are known, we can find their diſtance *AC*; for then we know *AB*, *CB*, the complements of the declinations, and the angle *B*, the difference of the right aſcenſions. Or, if *B* be the pole of the ecliptic, and we know the latitudes and longitudes of the ſtars, then we know *AB*, *BC*, the complements of the latitudes, and the angle *B*, the difference of the longitudes.

In like manner we may calculate all the caſes of Propoſitions 35, 36, 37, 38. and the ambiguity may, in moſt caſes, be taken away by ſome of the foregoing obſervations.

Ex. 7. *Given the latitude of a place* 52°. 12′ N. *the declination of the ſun* 15°. 30′ N. *and it's zenith diſtance* 108°; *to find the time.*

Here, $ZP = 37°. 48'$, $PS = 74°. 30'$, and $ZS = 108°$; to find the angle *P*. By Prop. 40. we find *P* by the following operation.

Ar. co. ſin. PS =	74°. 30′ - - - - -	0,0160895
Ar. co. ſin. PZ =	37. 48 - - - - -	0,2126054
		9,9790594
D =	36. 42	9,7655436
SZ =	108. 0	
		2)19,9732979
Sum =	144. 42	
		9,9866489
Diff. =	71. 18	
ſin. $\frac{1}{2}(SZ+D)$ =	72. 21	
ſin. $\frac{1}{2}(SZ-D)$ =	35. 39	
ſin. 75°. 51′. 56″ = $\frac{1}{2}P$		

Hence, the angle $P = 151°. 43'. 52''$, which converted into time, gives 10*h*. 44′. 4″ from apparent noon, that is, at 1*h*. 16′. 56″ in the morning, or 10*h*. 44′. 4″ in the evening, apparent time. Theſe are the times when twilight begins and ends.

By a like proceſs, whenever the ſun's altitude is known, the time may be found.

The laſt caſe may be computed by Art. 244. thus,

Log. coſ. PZ	$= 37°. 48'. 0''$ - -	9,8977123
——— PS	$= 74. 30. 0$ - -	9,4268988
——— M	$= 77. 48. 35,16$ - -	9,3246111
SZ	$= 108. 0. 0$	
$M+SZ$	$= 185. 48. 35,16$	
$M-SZ$	$= -30. 11. 24,84$	
		0,3010300
ſin. $\frac{1}{2}(M+SZ)$	$= 92. 54. 17,58$ - -	9,9994406
ſin. $\frac{1}{2}(M-SZ)$	$= -15. 5. 42,58$ - -	9,4156796
ar. co. ſin. PZ	$= 37. 48. 0$ - -	0,2126054
ar. co. ſin. PS	$= 74. 30. 0$ - -	0,0160895
coſ. P	$= 151. 43. 52$ - - -	9,9448451

As $\frac{1}{2}(M-SZ)$ is negative, it's ſine is negative (86), therefore coſ. P is negative, and conſequently P is the ſupplement of the angle which correſponds to log. coſ. 9,9448451.

PROP. XLVII.

Given two obſerved altitudes of the ſun, and the time between, with the change of declination in the interval of the obſervations, and the declination at the firſt; to find the latitude of the place.

254. Correct

FIG. 28.

254. Correct the obſerved altitudes for parallax and refraction, and for the dip of the horizon, if neceſſary; by which you obtain the true altitudes. Let Z be the zenith of the place, P the pole, S, s, the places of the ſun at the firſt and ſecond obſervations; join PS, Ps, ZS, Zs by great circles; take $Pt = PS$, and join St by a great circle; and draw Zx perpendicular to PS, Pr perpendicular to St, and tv perpendicular to Ss. Then we know PS, Ps, the complements of the declinations, st the change of declination, and the angle SPs correſponding to the interval of the obſervations. Now as st is very ſmall, stv may be conſidered as a plane triangle; alſo, the angle vst is very nearly $= Ptr = PSr$; and as $PS = Pt$, Pr biſects St, and the angle $rPS = \frac{1}{2}\, tPS$. Now

$$\text{ſin. } rS = \frac{\text{ſin. } SP \times \text{ſin. } rPS}{r} \ (212),$$

$$\text{cot. } rSP = \frac{\text{coſ. } SP \times \text{tan. } rPS}{r} \ (220).$$

$$tv = \text{ſin. } vst \times st \ (125) = \text{ſin. } rSP \times st, \ \text{rad.} = 1.$$

$$tSs = \text{coſec. } St \times tv \ (178) = \text{coſec. } St \times \text{ſin. } rSP \times st.$$

$$PSs = rSP \pm^{*} tSs.$$

$$sv = \text{coſ. } vst \times st \ (125) = \text{coſ. } rSP \times st.$$

$$Ss = St \pm^{\dagger} \text{coſ. } rSP \times st = 2rS \pm \text{coſ. } rSP \times st.$$

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* The ſign + is to be uſed when the north polar diſtance *increaſes*, and − when it *decreaſes*.

† If rSP (rtP) be *leſs* than 90°, and the north polar diſtance *increaſe*, or be *greater* than 90°, and the north polar diſtance *decreaſe*, the ſign + is to be uſed; otherwiſe, the ſign −.

Let $E = ZS - Ss$; then,

$$\text{ſin.}\ \tfrac{1}{2}\, ZSs = \sqrt{\frac{\text{ſin.}\ \frac{1}{2}(Zs+E) \times \text{ſin.}\ \frac{1}{2}(Zs-E)}{\text{ſin.}\ ZS \times \text{ſin.}\ Ss}} \quad (240).$$

$$ZSP = PSs \pm ZSs.$$

$$\text{tan.}\ Sx = \frac{\text{tan.}\ ZS \times \text{coſ.}\ ZSP}{r} \quad (219).$$

$$Px = PS \pm Sx.$$

$$\text{ſin.}\ lat. = \text{coſ.}\ PZ = \frac{\text{coſ.}\ ZS \times \text{coſ.}\ Px}{\text{coſ.}\ Sx} \quad (224).$$

Hence, let

P = north polar diſtance of the ſun at firſt obſervation;
$m = \frac{1}{2}$ interval of the obſervations, reduced to degrees;
Z = *true* zenith diſtance at the 1ſt obſervation;
z = ———————————— 2d ————
d = change of declination between the obſervations.

Then we have the following Rule;

Log. ſin. P + log. ſin. $m - 10$, = log. ſin. A.
Log. coſ. P + log. tan. $m - 10$, = log. cot. B.
$B \pm$[*]coſec. $2A \times$ ſin. $B \times d = C$.
$2A \pm$[†]coſ. $B \times d = D$.
$E = Z - D$.
$\frac{1}{2}\Big($ar. co. l. ſi. D + ar. co. l. ſi. Z + l. ſi. $\frac{1}{2}(z+E)$ + l. ſi. $\frac{1}{2}(z-E)\Big)$ = l. ſi. F.
$C \pm 2F = G$.
Log. coſ. G + log. tan. $Z - 10$, = log. tan. H.
$P \pm H = I$.
Log. coſ. I + log. coſ. Z + ar. co. log. coſ. $H - 10$, = log. ſin. *Lat.*

A is of the ſame affection as m.
B is leſs or greater than 90°, according as P is leſs, or greater than 90°, if m be leſs than 90°, which will generally be the caſe.
F is leſs than 90°.
H is of the ſame kind as G.

The computation of coſec. $2A \times$ ſin. $B \times d$, and coſ. $B \times d$, may be readily performed by logarithms, without any more openings of the Tables, except for d, whoſe logarithm will always be immediately found in the firſt four pages of the Tables; and five figures of decimals will generally be found ſufficient. In the *firſt* operation, 20 is rejected from the index, and in the *ſecond*, 10 is rejected, for the reaſon in Art. 50.

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[*] The ſign + is to be uſed when the north polar diſtance *increaſes*, and − when it *decreaſes*.

[†] If B be *leſs* than 90°, and the north polar diſtance *increaſe*, or be *greater* than 90°, and the north polar diſtance *decreaſe*, the ſign + is to be uſed; otherwiſe, the ſign −.

Ex. *Let* $P=76°$, $Z=48°.\ 26'.\ 48''$, $z=39°.\ 58'.\ 48''$, $m=11°.\ 15'$, *and* $d=82''$, *the north polar diſtance increaſing; to find the latitude of the place.*

P=76°. 0'. 0"	ſin.	9,9869041	
m=11. 15. 0	ſin.	9,2902357	
A=10. 54. 42	ſin.	9,2771398	
P=76. 0. 0	coſ.	9,3836752	
m=11. 15. 0	tan.	9,2986618	
B=87. 14. 42	cot.	8,6823370	
+0. 3. 40	82" - - - log.	1,91381	log. 1,91381
C=87. 18. 22	ſin.	9,99950	coſ. 8,68183
2A=21. 49. 24	coſec.	10,42975	4"...log. 0,59564
+0. 0. 4	220"=3'.40"....log.	2,34306	=coſ. D×d.
D=21. 49. 28	=coſec. 2A×ſin. B×d.		
Z=48. 26. 48			
E=Z−D=26. 37. 20			
z=39. 58. 48			
z+E=66. 36. 8	ar. co. ſin.	0,4297328	
z−E=13. 21. 28	ar. co. ſin.	0,1259019	
½(z+E)=33. 18. 4	ſin.	9,7396032	
½(z−E)= 6. 40. 44	ſin.	9,0655976	
	2)	19,3608355	
F=28. 37. 34	ſin.	9,6804177	
2F=57. 15. 8			
C=87. 18. 22			
G=C−2F=30. 3. 14	coſ.	9,9372945	
Z=48. 26. 48	tan.	10,0523771	
H=44. 19. 8	tan.	9,9896716	
P=76 0. 0			
I=P−H=31. 40. 52	coſ.	9,9299215	
Z=48. 26. 48	coſ.	9,8217211	
H=44. 19. 8	ar. co. coſ.	0,1454132	
Latitude=52. 5. 19	ſin.	9,8970558	

At land, the two obſervations may be made at the ſame place; but on board a ſhip in motion, the obſervations will be made in different places; in this caſe, the altitude at the ſecond obſervation muſt be reduced to what it would have been if the obſervation had been made at the place where the firſt altitude was taken; for the method of doing which, ſee my *Practical Aſtronomy*, page 50.

PROP. XLVIII.

To find the area of a ſpherical triangle.

255. Let ABC be the triangle; complete the circle AB, and produce BC, AC to meet at D, cutting the circle AB in Q, R. Then as $BR=180°$, and $CD=180°$ (159), we have $BC=DR$; for the ſame reaſon, $AC=DQ$; alſo, the angle $C=$ the angle D (173); therefore the triangles ABC, DQR are equal; let each be repreſented by a; and let b, c, d repreſent the reſpective triangles in which they are placed. Put $S=\frac{1}{4}$ of the ſurface of the ſphere, $r=$it's radius, $p=0{,}78539$ &c. then $pr^2=S$. Now by Art. 177. FIG. 35.

$$\begin{array}{l}
180^\circ : \angle BAC :: 2S : a+d \\
180^\circ : \angle ABC :: 2S : a+b \\
180^\circ : \angle ACB :: 2S : a+c \\
\hline
\therefore 180^\circ : \angle A+\angle B+\angle C :: 2S : 3a+b+d+c \\
\qquad\qquad\qquad\qquad\qquad :: 2S : 2a+2S \\
\qquad\qquad\qquad\qquad\qquad :: S : a+S;
\end{array}$$

hence, $a = S \times \frac{A+B+C-180^\circ}{180^\circ} = pr^2 \times \frac{A+B+C-180^\circ}{180^\circ}$.

* * *

ON THE VARIATION OF SPHERICAL TRIANGLES.

256. If the radius of a circle $=r$, $x=$ the ſine of an arc z, $y=$ it's coſine; then (ſee the *Principles of Fluxions*, Art. 46.) $r : y :: \dot{z} : \dot{x}$, and $r\dot{x}=y\dot{z}$; alſo, $r : x :: \dot{z} : -\dot{y}$, and $-r\dot{y}=x\dot{z}$; the $\dot{y}$ is here written negative, becauſe y decreaſes whilſt z increaſes. And what is true for the fluxions of quantities, will be nearly true for very ſmall increments.

PROP. XLIX.

Let ABC be a right angled ſpherical triangle, right angled at C; and let the angle A remain conſtant; then, $\dot{AB} : \dot{AC} :: \text{ſin. } 2AB : \text{ſin. } 2AC$. FIG. 29.

257. For draw *mr* perpendicular to *AC* and very near to *BC*, and *mn* parallel to *rC*; then conſidering the incremental triangle *Bmn* as rectilinear *,

$$\begin{aligned}
Bm : mn &:: r : \text{ſin. } B \ (125) \\
mn : rC &:: \text{coſ. } BC : r \ (178)
\end{aligned}$$

$$\begin{aligned}
\therefore Bm(\dot{AB}) : rC(\dot{AC}) &:: \text{coſ. } BC : \text{ſin. } B \\
&:: \frac{\text{coſ. } AB}{\text{coſ. } AC} : \frac{\text{ſin. } AC}{\text{ſin. } AB} \ (212) \\
&:: \text{ſin.} AB \times \text{coſ.} AB : \text{ſin.} AC \times \text{coſ.} AC \\
&:: \text{ſin. } 2AB : \text{ſin. } 2AC \ (97).
\end{aligned}$$

258. COR. When $\dot{AB} = \dot{AC}$, ſin. $2AB =$ ſin. $2AC$; therefore $AB + AC = 90°$.

PROP. L.

Let the hypothenuſe AB remain conſtant, and the two legs vary; then $\dot{BC} : \dot{AC} :: \text{tan. } AC : \text{tan. } BC$. FIG. 30.

259. For let *ABC* become *ABc*; then (212) cot. $d \times$ tan. $BC = r \times$ ſin. $Cd =$ (as *Cd* is very ſmall) $r \times Cd$; alſo (212), cot. $d \times$ tan. $Ac = r \times$ ſin. cd, or

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* In like manner, in the following propoſitions we ſuppoſe the incremental triangle to be rectilinear.

cot. $d \times$ tan. $AC = r \times cd$; hence, $\frac{Cd}{\text{tan. } BC} = \frac{\text{cot. } d}{r} = \frac{cd}{\text{tan. } AC}$, and cd $(\dot{BC})$: Cd $(\dot{AC})$ :: tan. AC : tan. BC.

Prop. LI.

FIG. 31. *In the right angled triangle ACB, let the baſe AC remain conſtant, and the angle A vary; then*

$\dot{AB}$: $\dot{BC}$ :: *tan.* BC : *tan.* AB
$\dot{BC}$: $\dot{A}$ :: *ſin.* AB : *ſin.* B.

260. For draw Am very near to AB, and mn perpendicular to AB; then Bn $(\dot{AB})$: Bm $(\dot{BC})$ :: coſ. B : r (125) :: tan. BC : tan. AB (219).

Alſo, Bm $(\dot{BC})$: mn :: r : ſin. B (128)
mn : BAm $(\dot{A})$ :: ſin. AB : r (178)

∴ $\dot{BC}$: $\dot{A}$ :: ſin. AB : ſin. B.

Prop. LII.

FIG. 32. *Let ABC be any ſpherical triangle, having the angle A and it's oppoſite ſide BC conſtant; then,*

$\dot{B}$: $\dot{AC}$ :: *tan.* B : *tan.* AC.
$\dot{C}$: $\dot{AB}$ :: *tan.* C : *tan.* AB.
$\dot{AB}$: $\dot{AC}$ :: *coſ.* C : *coſ.* B.
$\dot{B}$: $\dot{C}$ :: *coſ.* AC : *coſ.* AB.

261. Let BC become ns, on which let fall the perpendiculars Cr, Bm. Then mn is the decrement of the part Bv, and rs is the increment of the part vC; therefore $mn = rs$. Now (221) ſin. B : ſin. AC :: ſin. A : ſin. BC, and as the two laſt terms are conſtant, the fluxions of the two firſt terms muſt be in the ratio of the two laſt; hence (256),

$$\begin{aligned}
&\text{coſ.}\,B\times\dot{B} : \text{coſ.}\,AC\times\dot{AC} :: \text{ſin.}\,A : \text{ſin.}\,BC,\\
\text{or,}\quad &\dot{B} : \dot{AC} :: \text{ſin.}\,A\times\text{coſ.}\,AC : \text{ſin.}\,BC\times\text{coſ.}\,B\\
&\qquad :: \text{ſin.}\,B\times\text{coſ.}\,AC : \text{ſin.}\,AC\times\text{coſ.}\,B\text{*}\\
&\qquad :: \frac{\text{ſin.}\,B}{\text{coſ.}\,B} : \frac{\text{ſin.}\,AC}{\text{coſ.}\,AC}\\
&\qquad :: \text{tan.}\,B : \text{tan.}\,AC.
\end{aligned}$$

In like manner it appears, that $\dot{C} : \dot{AB}$ :: tan. C : tan. AB.

$$\begin{aligned}
\text{Again, } Bn\,(\dot{AB}) : nm, \text{ or } rs, &:: r : \text{coſ.}\,Bnm, \text{ or coſ.}\,B\ (125)\\
rs : Cs\,(\dot{AC}) &:: \text{coſ.}\,s, \text{ or coſ.}\,C, : r\ (125)\\
\hline
\therefore\ \dot{AB} : \dot{AC} &:: \text{coſ.}\,C : \text{coſ.}\,B.
\end{aligned}$$

Laſtly, if we take the ſupplemental triangle PQR to ABC, then as A and BC are conſtant in the triangle ABC, QR and the angle P are conſtant in the triangle PQR; and as the two latter are the ſupplements of the two former, the variations of the latter are the ſame as the variations of the former, and they have the ſame ſines, coſines and tangents reſpectively; but

FIG. 19.

---

* Becauſe ſin. A : ſin. BC :: ſin. B : ſin. AC (221).

but by the laſt caſe, $P\dot{R} : P\dot{Q} ::$ coſ. Q : coſ. R; therefore $\dot{B} : \dot{C} ::$ coſ. AC : coſ. AB.

PROP. LIII.

FIG. 33. *In any triangle ABC, let the angle A and the ſide AB remain conſtant; then,*

$$\begin{aligned}
A\dot{C} &: \dot{B} :: \text{ſin. } BC : \text{ſin. } C.\\
B\dot{C} &: \dot{B} :: \text{ſin. } BC : \text{tan. } C.\\
A\dot{C} &: B\dot{C} :: \text{rad.} : \text{coſ. } C.\\
A\dot{C} &: \dot{C} :: \text{tan. } BC : \text{ſin. } C.\\
\dot{B} &: \dot{C} :: \text{rad.} : \text{coſ. } BC.
\end{aligned}$$

262. Let BC become Br, and draw rm perpendicular to BC; then,

$$\begin{aligned}
&rC\,(A\dot{C}) : rm :: r : \text{ſin. } C \ (128)\\
&rm : rBC\,(\dot{B}) :: \text{ſin. } Br, \text{ or ſin. } BC, : r \ (178)\\
\therefore\ &A\dot{C} : \dot{B} :: \text{ſin. } BC : \text{ſin. } C.
\end{aligned}$$

$$\begin{aligned}
\text{Again, } &Cm\,(B\dot{C}) : mr :: \text{ſin. } mrC, \text{ or coſ. } C, : \text{ſin. } C \ (128)\\
&mr : rBC\,(\dot{B}) :: \text{ſin. } BC : r \ (178)\\
\therefore\ &B\dot{C} : \dot{B} :: \text{coſ. } C \times \text{ſin. } BC : \text{ſin. } C \times r\\
&:: \text{ſin. } BC : \text{tan. } C.
\end{aligned}$$

Next, $rC\,(A\dot{C}) : mC\,(B\dot{C}) :: r :$ coſ. C (125).

FIG. 19. Take now the ſupplemental triangle PQR to ABC; then as A and AB are conſtant in ABC, QR and R are conſtant in PQR; therefore by the laſt caſe but one in this propoſition, $\dot{Q} : P\dot{Q} ::$ tan. P : ſin. PQ, that is, $A\dot{C} : \dot{C} ::$ tan. BC : ſin. C.

Laſtly, as $\dot{B} : A\dot{C} :: \text{ſin.}\ C : \text{ſin.}\ BC$
and $A\dot{C} : \dot{C} :: \text{tan.}\ BC : \text{ſin.}\ C$

$$\therefore \dot{B} : \dot{C} :: \text{tan.}\ BC : \text{ſin.}\ BC$$
$$:: r : \text{coſ.}\ BC \ (78)$$

PROP. LIV.

In any triangle ABC, let the two ſides AB, AC remain conſtant; then, FIG. 34.

$\dot{A} : B\dot{C} :: \text{coſec.}\ C : \text{ſin.}\ AC$, *or*, $:: \text{coſec.}\ B : \text{ſin.}\ AB$.
$\dot{A} : \dot{B} :: r \times \text{ſin.}\ BC : \text{ſin.}\ AC \times \text{coſ.}\ ACB$.
$\dot{C} : \dot{B} :: \text{tan.}\ C : \text{tan.}\ B$.
$B\dot{C} : \dot{B} :: \text{ſin.}\ BC : \text{cot.}\ C$.

263. Let ABC become ABm, taking $Am = AC$; join Bm, and draw Cr perpendicular to it; then,

$$\dot{A} : Cm :: r : \text{ſin}\ AC \ (178)$$
$$Cm : mr\ (B\dot{C}) :: r : \text{ſin.}\ rCm, \text{ or ſin.}\ ACB \ (128)$$

$$\therefore \dot{A} : B\dot{C} :: r^2 : \text{ſin.}\ AC \times \text{ſin.}\ ACB$$
$$:: \frac{r^2}{\text{ſin.}\ ACB}, \text{ or } (83) \text{ coſec.}\ ACB, : \text{ſin.}\ AC;$$
$$\text{or, } :: r^2 : \text{ſin.}\ AB \times \text{ſin.}\ B^{*}$$
$$:: \frac{r^2}{\text{ſin.}\ B}, \text{ or } (83) \text{ coſec.}\ B, : \text{ſin.}\ AB.$$

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* For (221) ſin. AB : ſin. AC :: ſin. ACB : ſin. B, therefore ſin. $AB \times$ ſin. $B =$ ſin. $AC \times$ ſin. ACB.

Again, $\dot{A} : Cm :: r : \text{ſin. } AC$ (178)
$Cm : Cr :: r : \text{ſin. } Cmr$ (128), or coſ. AmB, or coſ. ACB
$Cr : \dot{B} :: \text{ſin. } BC : r$ (178)

$$\therefore \dot{A} : \dot{B} :: r \times \text{ſin. } BC : \text{ſin. } AC \times \text{coſ. } ACB$$

Next, ſin. C : ſin. B :: ſin. AB : ſin. AC (221), and as the two laſt terms are conſtant, the fluxions of the two firſt terms muſt be in the ratio of the two laſt; hence (256),

$\text{Coſ. } C \times \dot{C} : \text{coſ. } B \times \dot{B} :: \text{ſin. } AB : \text{ſin. } AC$
or, $C : \dot{B} :: \text{coſ. } B \times \text{ſin. } AB : \text{coſ. } C \times \text{ſin. } AC$
$:: \text{coſ. } B \times \text{ſin. } C : \text{coſ. } C \times \text{ſin. } B$ *
$:: \text{tan. } C : \text{tan. } B.$

Laſtly, as $B\dot{C} : \dot{A} :: \text{ſin. } AC \times \text{ſin. } ACB : r^2$
and $\dot{A} : \dot{B} :: r \times \text{ſin. } BC : \text{ſin. } AC \times \text{coſ. } C$

$\therefore B\dot{C} : \dot{B} :: \text{ſin. } BC \times \text{ſin. } C : r \times \text{coſ. } C$
$:: \text{ſin. } BC : \text{cot. } C$ (81)

264. Cor. By Art. 248. the tang. of $C = \dfrac{r^2 \times \text{ſin. } B}{\text{ſin. } BC \times \text{cot. } AB - \text{coſ. } BC \times \text{coſ. } B}$, therefore cot. C $\left(= \dfrac{r^2}{\text{tan. } C}\right) = \dfrac{\text{ſin. } BC \times \text{cot. } AB - \text{coſ. } BC \times \text{coſ. } B}{\text{ſin. } B}$; ſubſtitute this for cot. C in the laſt proportion, and we get $\dot{B} = B\dot{C} \times \left(\dfrac{\text{cot. } AB}{\text{ſin. } B} - \dfrac{\text{cot. } BC}{\text{tan. } B}\right)$. If B repre-

ſent

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* Becauſe ſin. AB : ſin. AC :: ſin. C : ſin. B (221).

ſent the pole of the equator, A the zenith, C the ſun, this equation gives the correction for finding the time of apparent noon, from equal altitudes of the ſun.

Prop. LV.

In any triangle ABC, let the two angles B, C, remain conſtant; then,

$\dot{BC} : \dot{A}$ :: *coſec.* AB : *ſin.* B, *or*, :: *coſec.* AC : *ſin.* C.
$\dot{BC} : \dot{AC}$ :: $r \times$ *ſin.* A : *ſin.* $B \times$ *coſ.* AB.
$\dot{AB} : \dot{AC}$ :: *tan.* AB : *tan.* AC.
$\dot{A} : \dot{AC}$ :: *ſin.* A : *cot.* AB.

265. Take the ſupplemental triangle PQR to ABC, then PR and PQ are conſtant; therefore by Prop. 54. FIG. 19.

$\dot{P} : \dot{QR}$ :: coſec. R : ſin. PR, or, :: coſec. Q : ſin. PQ, or, $\dot{BC} : \dot{A}$ :: coſec. AB : ſin. B, or, :: coſec. AC : ſin. C.

Again, $\dot{P} : \dot{Q}$ :: $r \times$ ſin. QR : ſin. $PR \times$ coſ. R,
or, $\dot{BC} : \dot{AC}$ :: $r \times$ ſin. A : ſin. $B \times$ coſ. AB.

Alſo, $\dot{R} : \dot{Q}$ :: tan. R : tan. Q,
or, $\dot{AB} : \dot{AC}$ :: tan. AB : tan. AC.

Laſtly, $\dot{QR} : \dot{Q}$ :: ſin. QR : cot. R,
or, $\dot{A} : \dot{AC}$ :: ſin. A : cot. AB.

266. Theſe theorems are of great uſe in Aſtronomy, where, having given the variation of one quantity, it is frequently required to find the cotemporary variation of another which depends upon it. For inſtance, in Prop. 54. If A repreſent the pole of the equator, B the zenith, C the ſun, then $\dot{A}$ repreſents the variation of the hour-angle, and $B\dot{C}$ the correſponding variation of the altitude; if therefore one be given, the other may be found from the proportion between theſe quantities. Alſo, $\dot{B}$ repreſents the correſponding variation of the azimuth, which is therefore known when $\dot{A}$ is given. If $BC = 90°$, and $B\dot{C}$ = the ſun's diameter, then A (reduced into time) gives the time in which the ſun aſcends above the horizon. If $B\dot{C} = 33'$, then $\dot{A}$ gives the time by which the riſing of the ſun is accelerated by refraction. It is however ſometimes neceſſary to reduce the ratios here given, by ſubſtituting for ſome, or all of the quantities, others which are equal to them, in order to bring them to a more convenient form for the calculation, an inſtance of which is given in Cor. Prop. 54. The nature of the reduction muſt depend upon the form to which it may be proper to reduce the ratio.

267. If we ſuppoſe the ſides of the triangle ABC to be diminiſhed without limit, it will approach to a rectilinear triangle as the limit; and inſtead of the ſines and tangents of the ſides, we may ſubſtitute the ſides themſelves. For inſtance, if ABC be a rectilinear triangle, when two angles, B, C, remain conſtant; we have (265), $A\dot{B} : A\dot{C} :: AB : AC$. And if the two ſides AB, AC, remain conſtant, we have (263),

(263), $\dot{A} : \dot{B} :: r \times CB :$ coſ. $C \times AC$.' In this manner, the variations of plane triangles may be deduced from thoſe of ſpherical; the variations however of the former may be immediately deduced from the triangles, in a manner, exactly ſimilar to that by which we deduced the latter.

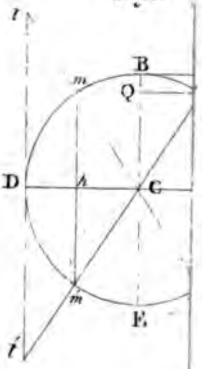
ERRATA AND ADDENDA.

PAGE 64. l. 8. from bottom *for* next, *read* 15th.
l. 3. from bottom *for* 50, *read* 52.
85, l. 7. from bottom, *after* Tables, *inſert as follows*:

Let $a^\circ.\ b'$ be an arc, whoſe logarithm (l) is next *leſs* than the given logarithm (L), if it be that of a *ſine, tangent,* or *ſecant,* or next *greater,* if it be that of a *coſine, cotangent,* or *coſecant;* and let d be the difference of the logarithms correſponding to $1''$; then $d : L \sim l :: 1'' : c''$, and $a^\circ.\ b'.\ c''$ is the arc correſponding to the given logarithm (L). On the contrary, given the arc $a^\circ.\ b'.\ c''$ to find the logarithm L. Let l be the logarithm of $a^\circ.\ b'$; then $1'' : c'' :: d : D$, and $L + D$ is the required logarithm, if it be that of a *ſine, tangent,* or *ſecant,* and $L - D$ is the logarithm, if it be that of a *coſine, cotangent,* or *coſecant.*

7.

Fig. 1



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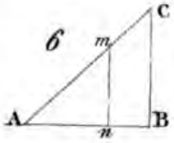

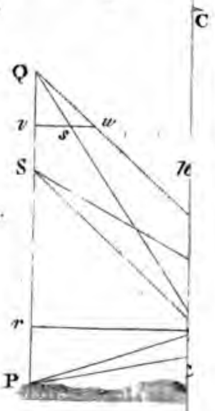

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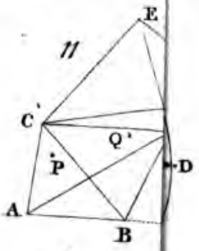

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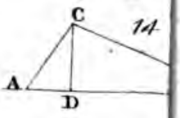
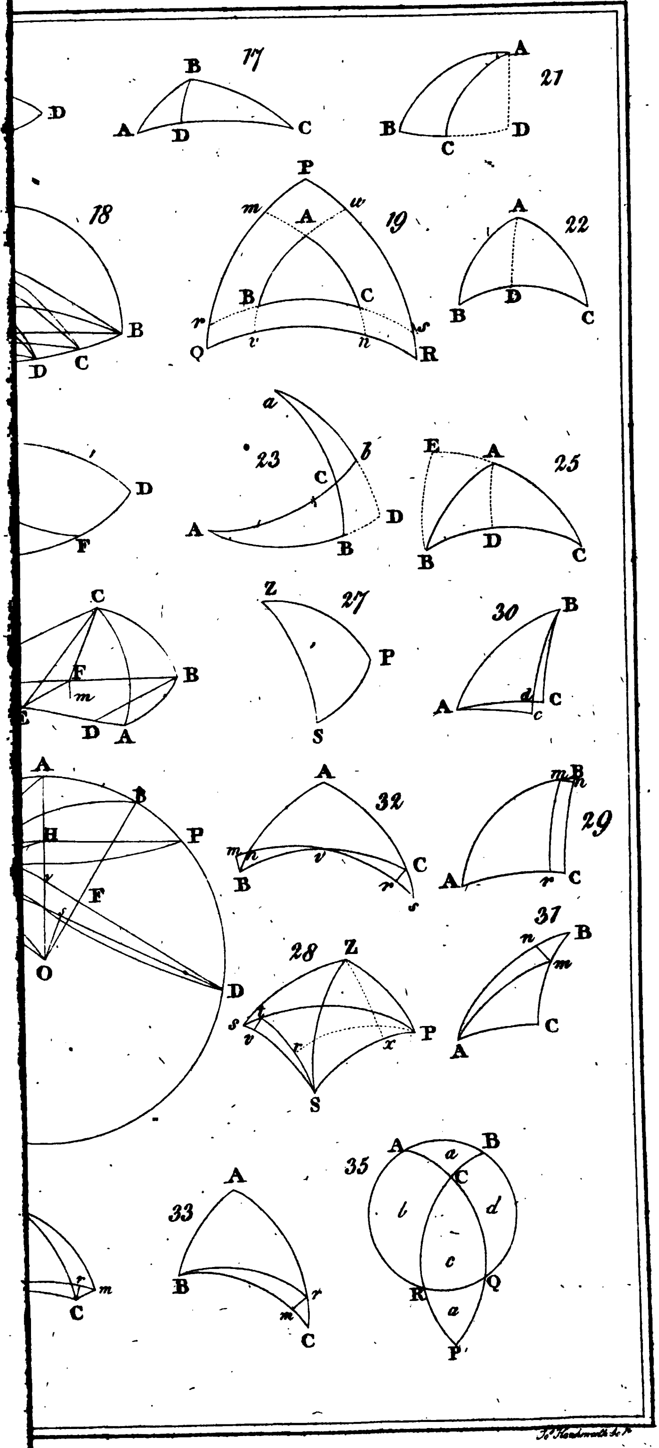


Pl. 2.



Jo. Hochmuth sc. P.

ERRATA TO TRIGONOMETRY.

PAGE 24. l. 4. *for* ar. co. U, *read* ar. co. log. U.

72. l. laſt but one, *for* 9,999998, *read* 9,9999998.

138. l. 8. *for* ſin. $BAE = \frac{r \times \text{ſin. } BE}{\text{ſin. } AB}$, *read* coſ. $BAE = \frac{\text{coſ. } BE \times \text{ſin. } ABE}{r}$. This Propoſition, however, was inſerted by miſtake, it being the ſame as Prop. 38.

161. l. 10. *after* AC *put* log. coſ. BC.

168. l. 13. *for* coſ. $D \times d$, *read* coſ. $B \times d$.

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